Conformal Field Theory Dual To $f(T)$ Black Holes

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Abstract

We extend the black hole holography to the case of an asymptotically anti-de Sitter (AdS) rotating charged black holes in $f(T) = T + \alpha T^2$ gravity, where α is a constant. We find that the scalar wave radial equation at the near-horizon region implies the existence of the 2D conformal symmetries. We show that choosing proper central charges for the dual CFT, we produce exactly the macroscopic Bekenstein-Hawking entropy from the microscopic Cardy entropy for the dual CFT. These observations suggest that the rotating charged AdS black hole in $f(T)$ gravity is dual to a 2D CFT at finite temperatures.[1]

Keywords: $f(T)$-gravity, black hole holography, conformal field theory
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1. INTRODUCTION

The idea of AdS/CFT correspondence [1] was extended to the case of extremal rotating black holes, namely, the Kerr/CFT correspondence which was proposed by Guica et al. [2]. The correspondence states that the physics of the extremal Kerr black holes, which are rotating with maximum angular velocity, can be described by a 2D CFT, living on the near-horizon region of the black holes. The correspondence was established by showing that one can microscopically reproduces the Bekenstein-Hawking entropy, using the CFT Cardy entropy formula. As one would expect, the Kerr/CFT correspondence is not only a peculiar property of extremal black holes but also non-extremal Kerr black holes. However, at the near-horizon region of the non-extremal Kerr black holes, one cannot indicate any conformal symmetries. In other words, the conformal symmetries are not the symmetries of the non-extremal Kerr black hole geometry (as they are for the case of the extremal Kerr black holes). However, it turns out that the "hidden" conformal symmetries can be revealed by looking at the solution space of the radial part of the Klein-Gordon equation, for a massless scalar probe in the near-horizon region of the Kerr black holes [3]. In this case, the radial equation, can be written as the $SL(2,R)_L \times SL(2,R)_R$ Casimir eigen-equation. Subsequently, the Kerr/CFT correspondence can be established by matching the microscopic CFT Cardy entropy to the macroscopic Bekenstein-Hawking entropy of the Kerr black holes with general angular momentum and mass parameters. The correspondence has been studied for several black holes solutions, for instance, in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

The usual theory of gravity, based on the Riemannian geometry, has been extended through several gravity theories. One of them is the teleparallel gravity (TG) theory, where the Ricci scalar R, is replaced by the teleparallel torsion scalar T. Moreover, the TG has been generalized to $f(T)$ gravity by replacing the torsion scalar T, with an arbitrary function of T, such as $f(T)$. In [15], the authors find an asymptotically rotating charged AdS black hole solution, in quadratic $f(T)$ gravity, where $f(T) = T + \alpha T^2$. A very natural question to be asked, is "Do we have any $f(T)$/Hidden CFT correspondence?", that we address in this article.

The outline of this presentation is organized as follows. In Section 2, we review the $f(T)$-Maxwell gravity theory as well as the rotating charged AdS black hole solutions, and its thermodynamics aspects. We also consider the massless Klein-Gordon wave equation, in the background of the rotating charged AdS black holes, in quadratic $f(T)$ gravity. In Section 3, we study the radial wave equation in the near-horizon region of the black holes, and rewrite it as the $SL(2,R)_L \times SL(2,R)_R$ Casimir equation. We also find the central charges of the dual CFT by matching the Cardy entropy for the dual CFT to the macroscopic Bekenstein-Hawking entropy. Therefore, we present evidence that the rotating charged AdS black holes in quadratic $f(T)$ gravity, can be considered holographically dual to the CFT. In the final section, we summarize our results and address some future works. We use the Planck units, in which $c = G = \hbar = k_B = 1$.

2. QUADRATIC $f(T)$ GRAVITY AND THE BLACK HOLE SOLUTIONS

The basic variables in TG are tetrad fields ${e_a}^\mu$, where $a = (0,1,2,3)$ is the index of internal space and $\mu = (0,1,2,3)$ is the index of spacetime. The tetrad fields satisfy

$$ {e_a}^\mu {e^a}_\nu = \delta^\mu_\nu, \quad {e_a}^\mu {e^b}_\mu = \delta^b_a. \tag{1}$$

[1] This presentation is entirely based on the published papers: M. Ghezelbash, *JHEP* **0908**, 045 (2009); C. Bernard and M. Ghezelbash, *Phys. Rev.* **D 101**, 024020 (2020); B. Fahim and M. Ghezelbash, *Int. J. Mod. Phys.* **A** to appear (2024).

The tetrad fields are related to the spacetime metric and its inverse

$$g_{\mu\nu} = \eta_{ab}e^{a}{}_{\mu}e^{b}{}_{\nu}, \quad g^{\mu\nu} = \eta^{ab}e_{a}{}^{\mu}e_{b}{}^{\nu}, \tag{2}$$

respectively, where $\eta_{ab} = \mathrm{diag}(-,+,+,+)$ is the metric of 4D Minkowski spacetime. Also, it can be shown that $e = \det(e_{a\mu}) = \sqrt{-g}$, where g is the determinant of the metric. In TG, we use the Weitzenbock connection

$$W^{\alpha}{}_{\mu\nu} = e_{a}{}^{\alpha}\partial_{\nu}e^{a}{}_{\mu} = -e^{a}{}_{\mu}\partial_{\nu}e_{a}{}^{\alpha}, \tag{3}$$

to define the covariant derivative, by

$$\nabla_{\nu}e_{a}{}^{\mu} = \partial_{\nu}e_{a}{}^{\mu} + W^{\mu}{}_{\rho\nu}e_{a}{}^{\rho} = 0. \tag{4}$$

The Weitzenbock connection is curvature-free, but it has a non vanishing torsion

$$T^{\alpha}{}_{\mu\nu} = W^{\alpha}{}_{\nu\mu} - W^{\alpha}{}_{\mu\nu} = e_{i}{}^{\alpha}\left(\partial_{\mu}e^{i}{}_{\nu} - \partial_{\nu}e^{i}{}_{\mu}\right). \tag{5}$$

We define the torsion scalar by

$$T = T^{\alpha}{}_{\mu\nu}S_{\alpha}{}^{\mu\nu}, \tag{6}$$

where the superpotential tensor is

$$S_{\alpha}{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_{\alpha} + \delta^{\mu}_{\alpha}T^{\beta\nu}{}_{\beta} - \delta^{\nu}_{\alpha}T^{\beta\mu}{}_{\beta}\right). \tag{7}$$

We note that the Contortion tensor $K_{\alpha\mu\nu}$ is given by

$$K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\nu\alpha\mu} + T_{\alpha\mu\nu} - T_{\mu\alpha\nu}\right). \tag{8}$$

We consider a four-dimensional rotating charged AdS black hole solution in $f(T)$-Maxwell theory with a negative cosmological constant where

$$f(T) = T + \alpha T^{2}. \tag{9}$$

The dimensional negative parameter α is the coefficient of the quadratic term of the scalar torsion. The action of the $f(T)$-Maxwell theory in 4D, for an asymptotically AdS spacetimes, is given by

$$\mathcal{S} = \frac{1}{2K}\int d^{4}x|e|\left(f(T) - 2\Lambda - F\wedge^{*}F\right), \tag{10}$$

where $\Lambda = -3/l^{2}$ is the 4D cosmological constant, l is the length scale of AdS spacetime. The constant K in (10) is related to the 4D Newton's gravitational constant G_4, by $K = 2\Omega_2 G_4$, where $\Omega_2 = 2\pi^{3/2}/\Gamma(3/2)$, is the volume of 2D unit sphere, and $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$.

In the action (10), $F = d\tilde{\Phi}$, where $\tilde{\Phi} = \tilde{\Phi}_{\mu}dx^{\mu}$ is the gauge potential one-form.

Varying action (10) with respect to the tetrad fields and the Maxwell potential Φ_{μ}, one finds the field equations for gravity

$$S_{\mu}{}^{\rho\nu}\partial_{\rho}Tf''(T) + \left[e^{-1}e^{a}{}_{\mu}\partial_{\rho}\left(ee_{\alpha}{}^{\alpha}S_{\alpha}{}^{\rho\nu}\right) - T^{\alpha}{}_{\lambda\mu}S_{\alpha}{}^{\nu\lambda}\right]f'(T), \tag{11}$$

$$-\frac{\delta^{\nu}_{\mu}}{4}\left(f(T) + \frac{6}{l^{2}}\right) = -\frac{K}{2}\mathcal{T}^{\nu}_{em\,\mu}, \tag{12}$$

and the Maxwell's equations

$$\partial_{\nu}\left(\sqrt{-g}F^{\mu\nu}\right) = 0, \tag{13}$$

respectively.

In equation (12), $\mathcal{T}^{\nu}_{em\,\mu} = F_{\mu\alpha}F^{\nu\alpha} - 1/4\delta_{\mu}{}^{\nu}F_{\alpha\beta}F^{\alpha\beta}$, is the energy-momentum tensor of the electromagnetic field. The rotating charged AdS black hole solution, is given by [15]

$$ds^{2} = -A(r)(\Xi dt - \Omega d\phi)^{2} + \frac{dr^{2}}{B(r)} + \frac{r^{2}}{l^{4}}\left(\Omega dt - \Xi l^{2}d\phi\right)^{2} + \frac{r^{2}}{l^{2}}dz^{2}, \tag{14}$$

where the range of coordinates are given by $-\infty < t, z < \infty$, $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$.

In metric (14), we have

$$A(r) = r^2\Lambda_{\text{eff}} - \frac{M}{r} + \frac{3Q^2}{2r^2} + \frac{2Q^3\sqrt{6|\alpha|}}{6r^4}, \tag{15}$$

$$B(r) = A(r)\beta(r), \tag{16}$$

$$\beta(r) = \left(1 + Q\sqrt{6|\alpha|}/r^2\right)^{-2}, \tag{17}$$

$$\Xi = \sqrt{1 + \frac{\Omega^2}{l^2}}, \tag{18}$$

where $\Lambda_{\text{eff}} = \frac{1}{36|\alpha|}$, and M, Q, and Ω are the mass parameter, the charge parameter, and the rotation parameter, respectively. The parameter α cannot be zero, since the effective cosmological constant Λ_{eff}, and the metric functions $A(r)$ and $B(r)$ become singular. The The gauge potential one-form $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(r) = -\Phi(r)(\Omega d\phi - \Xi dt), \tag{19}$$

where $\Phi(r) = \frac{Q}{r} + \frac{Q^2\sqrt{6|\alpha|}}{3r^3}$. We note that the torsion scalar T, for the black hole solution (14), is given by

$$T(r) = \frac{4A'(r)B(r)}{rA(r)} + \frac{2B(r)}{r^2}, \tag{20}$$

where $A(r)$ and $B(r)$, in equations (15) and (16).

We notice that setting the rotational parameter $\Omega = 0$, we find the static charged black hole configuration [16]. Moreover, turning off the mass parameter M and Q, the metric (14) reduces to, the 4D AdS metric in an uncommon coordinate system.

The horizons of the black holes are the positive roots of $A = 0$, among which, the outer one is denoted by r_+. The non-vanishing components of the contravariant metric tensor are given by

$$g^{tt} = \frac{l^4\left(A(r)\Omega^2 - r^2\Xi^2\right)}{A(r)r^2(\Xi^2 l^2 - \Omega^2)^2}, \quad g^{rr} = B(r), \quad g^{zz} = \frac{l^2}{r^2}, \tag{21}$$

$$g^{\phi\phi} = \frac{A(r)\Xi^2 l^4 - r^2\Omega^2}{A(r)r^2(\Xi^2 l^2 - \Omega^2)^2}, \quad g^{t\phi} = g^{\phi t} = \frac{\Xi\Omega l^2\left(A(r)l^2 - r^2\right)}{A(r)r^2(\Xi^2 l^2 - \Omega^2)^2}, \tag{22}$$

and the determinant of the metric is

$$\sqrt{-g} = \sqrt{\frac{A(r)}{B(r)}}\frac{r^2\left(\Xi^2 l^2 - \Omega^2\right)}{l^3}. \tag{23}$$

We find the area of the outer horizon $\mathcal{A}$, by setting $dt = dr = 0$ in the metric (14), and find

$$\mathcal{A} = \int_0^{2\pi} d\phi \int_0^{L} dz\sqrt{-g|_{dt=dr=0}} = \frac{2\pi r_+^2 \Xi L}{l}. \tag{24}$$

The entropy of black holes in $f(T)$ gravity becomes [17]

$$S = \frac{f'(T)\mathcal{A}}{4} = \frac{\pi r_+^2 \Xi L}{2l}(1 + 2\alpha T). \tag{25}$$

Using equation (20) in equation (25), we find the entropy of black hole (14), is given by

$$S = \frac{\pi\Xi L\left(7{r_+}^6 + 9\sqrt{6|\alpha|}Q{r_+}^4 + 18M|\alpha|{r_+}^3 - 54Q^2|\alpha|{r_+}^2 - 42\sqrt{6|\alpha|}^3 Q^3\right)}{9l\left(Q\sqrt{6|\alpha|} + {r_+}^2\right)^2}. \tag{26}$$

3. SCALAR PROBE FIELD EQUATIONS

We consider a massless scalar field Φ, in the background of the rotating charged AdS black holes (14), in quadratic $f(T)$ gravity. The scalar wave equation is given by [18, 19]

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = 0. \tag{27}$$

We consider the following ansatz for the scalar field

$$\Phi(t,r,z,\phi) = e^{-i\omega t + ikz + im\phi} R(r), \tag{28}$$

where ω is the frequency of the scalar field, m is the azimuthal harmonic index, and k is the wave number. Substituting equations (22), (23), and (28) into equation (27), we find the radial equation

$$B(r)\frac{d^2R(r)}{dr^2} + \left(rB(r)\frac{dA(r)}{dr} + rA(r)\frac{dB(r)}{dr} + 4A(r)B(r)\right)\frac{dR(r)}{dr} + V(r)R(r) = 0, \tag{29}$$

where the potential $V(r)$, is given by

$$V(r) = \frac{r^2\left(\Xi l^2\omega - \Omega m\right)^2 - A(r)l^2\left\{k^2 l^4 \Xi^4 + k^2\Omega^4 + l^2\left[m^2\Xi^2 - 2m\Xi\Omega\omega + \Omega^2\left(\omega^2 - 2\Xi^2 k^2\right)\right]\right\}}{A(r)r^2(\Xi^2 l^2 - \Omega^2)^2}. \tag{30}$$

In the near-horizon region, we expand the metric function $A(r)$ as a quadratic polynomial in $(r - r_+)$, such as

$$A(r) \simeq K\left(r - r_+\right)\left(r - r_*\right), \tag{31}$$

where

$$K = 15{r_+}^4\Lambda_{\text{eff}} - 3Mr_+ + \frac{3Q^2}{2}, \tag{32}$$

$$r_* = r_+ - \frac{2r_+\left(2{r_+}^4\Lambda_{\text{eff}} - Mr_+ + Q^2\right)}{10{r_+}^4\Lambda_{\text{eff}} - 2Mr_+ + Q^2}. \tag{33}$$

We note that r_* is not necessarily any of the black hole horizons. In the near-horizon region, we consider the low-energy limit for the scalar fields, where $r_+ \ll \frac{1}{\omega}$. Moreover, we consider a limit where the outer horizon r_+ is very close to r_*, in which, $|r_+ - r_*| \ll r_+$. Using these two approximations, we find that the radial equation (29) simplifies to

$$\frac{d}{dr}\left\{\left(r - r_+\right)\left(r - r_*\right)\frac{d}{dr}R(r)\right\} + \left[\left(\frac{r_+ - r_*}{r - r_+}\right)\mathcal{A} + \left(\frac{r_+ - r_*}{r - r_*}\right)\mathcal{B} + \mathcal{C}\right]R(r) = 0, \tag{34}$$

where the constants $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by

$$\mathcal{A} = \frac{\mathcal{D}m^2 + \mathcal{E}m\omega}{K^2{r_+}^2{r_*}^3(\Xi^2 l^2 - \Omega^2)^2(r_+ - r_*)^2\beta} + \frac{\mathcal{F}\omega^2}{K{r_+}^2(\Xi^2 l^2 - \Omega^2)^2(r_+ - r_*)^2\beta} - C_1, \tag{35}$$

$$\mathcal{B} = \frac{\mathcal{G}m^2 + \mathcal{I}m\omega}{K^2{r_+}^3 r_*(\Xi^2 l^2 - \Omega^2)^2(r_+ - r_*)^2\beta} + \frac{\mathcal{J}\omega^2}{K{r_+}^2(\Xi^2 l^2 - \Omega^2)^2(r_+ - r_*)^2\beta} + C_2, \tag{36}$$

$$\mathcal{C} = -\frac{2m\Omega\left(\Xi l^2\omega - \Omega m/2\right)^2\left({r_+}^2 + r_+ r_* + {r_*}^2\right)}{(\Xi^2 l^2 - \Omega^2)^2 K^2 {r_+}^3 {r_*}^3\beta}. \tag{37}$$

In equations (35)–(37), the constants C_1 and C_2 are given by $C_1 = C_2 = k^2 l^2/K\beta(r_+ - r_*)^2{r_+}^2$, and

$$\mathcal{D} = \Omega^2\left({r_+}^3 + 2{r_+}^2 r_* + 3{r_*}^2 r_+\right) - l^4\Xi^2{r_*}^3 K, \tag{38}$$

$$\mathcal{E} = 2\Omega\Xi l^2\left(Kl^2{r_*}^3 - {r_+}^3 - 2{r_+}^2 r_* - 3r_+{r_*}^2\right), \tag{39}$$

$$\mathcal{F} = -l^4\Omega^2, \tag{40}$$

$$\mathcal{G} = K\Xi^2 l^4 r_+ r_* - \Omega^2\left(3{r_+}^2 + 2r_+ r_* + {r_*}^2\right), \tag{41}$$

$$\mathcal{I} = 2l^2\Xi\Omega\left[3{r_+}^2 - r_+ r_*\left(Kl^2 - 2\right) + {r_*}^2\right], \tag{42}$$

$$\mathcal{J} = l^4\Omega^2. \tag{43}$$

4. HOLOGRAPHY FOR ROTATING $f(T)$ BLACK HOLES

To find the existence of the possible hidden symmetry, we introduce the following conformal coordinates ω^+, ω^-, and y, in terms of the black hole coordinates t, r, and ϕ

$$\omega^+ = \sqrt{\frac{r - r_+}{r - r_*}}e^{2\pi T_R\phi + 2n_R t}, \tag{44}$$

$$\omega^- = \sqrt{\frac{r - r_+}{r - r_*}}e^{2\pi T_L\phi + 2n_L t}, \tag{45}$$

$$y = \sqrt{\frac{r_+ - r_*}{r - r_*}}e^{\pi(T_L + T_R)\phi + (n_L + n_R)t}, \tag{46}$$

where T_L, T_R, n_L, and n_R are constants. We also define the sets of *local* vector fields

$$H_1 = i\partial_+, \tag{47}$$
$$H_0 = i\left(\omega^+\partial_+ + \frac{1}{2}y\partial_y\right), \tag{48}$$
$$H_{-1} = i\left(\omega^{+2}\partial_+ + \omega^+ y\partial_y - y^2\partial_-\right), \tag{49}$$

as well as

$$\bar{H}_1 = i\partial_-, \tag{50}$$
$$\bar{H}_0 = i\left(\omega^-\partial_- + \frac{1}{2}y\partial_y\right), \tag{51}$$
$$\bar{H}_{-1} = i\left(\omega^{-2}\partial_- + \omega^- y\partial_y - y^2\partial_+\right). \tag{52}$$

The vector fields (47)–(52) obey the $sl(2,R)_L \times sl(2,R)_R$ algebra, as

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1}, \quad [H_{-1}, H_1] = -2iH_0, \tag{53}$$
$$[\bar{H}_0, \bar{H}_{\pm 1}] = \mp i \bar{H}_{\pm 1}, \quad [\bar{H}_{-1}, \bar{H}_1] = -2i\bar{H}_0. \tag{54}$$

The quadratic Casimir operators of $sl(2,R)_L \times sl(2,R)_R$ algebra, are given by

$$\mathcal{H}^2 = \bar{\mathcal{H}}^2 = -H_0^2 + \frac{1}{2}\left(H_1 H_{-1} + H_{-1}H_1\right) = \frac{1}{4}\left(y^2\partial_y^2 - y\partial_y\right) + y^2\partial_+\partial_-. \tag{55}$$

We notice that the Casimir operators (55), can be rewritten in terms of (t, r, ϕ) coordinates, as

$$\begin{aligned}\mathcal{H}^2 = &\left(r - r_+\right)\left(r - r_*\right)\frac{\partial^2}{\partial r^2} + \left(2r - r_+ - r_*\right)\frac{\partial}{\partial r} \\ &+ \left(\frac{r_+ - r_*}{r - r_*}\right)\left[\left(\frac{n_L - n_R}{4\pi G}\partial_\phi - \frac{T_L - T_R}{4G}\partial_t\right)^2 + C_2\right] \\ &- \left(\frac{r_+ - r_*}{r - r_+}\right)\left[\left(\frac{n_L + n_R}{4\pi G}\partial_\phi - \frac{T_L + T_R}{4G}\partial_t\right)^2 - C_1\right],\end{aligned} \tag{56}$$

where $G = n_L T_R - n_R T_L$.

The Casimir operator (56) reproduces the radial equation (34), by choosing the right and left temperatures, as

$$T_R = \frac{r_+ K\left(r_+ - r_*\right)\left(\Xi^2 l^2 - \Omega^2\right)\sqrt{\beta r_+ r_* \delta}}{4\pi\delta}, \tag{57}$$
$$T_L = \frac{r_+ K\left(\Xi^2 l^2 - \Omega^2\right)\left[{r_+}^4 + 2{r_+}^3 r_* + 6{r_+}^2{r_*}^2 - 2{r_*}^3 r_+\left(Kl^2 - 1\right) + {r_*}^4\right]\sqrt{\beta r_+ r_* \delta}}{4\pi(r_+ + r_*)^3\delta}, \tag{58}$$

respectively, and

$$n_R = 0, \tag{59}$$
$$n_L = \frac{{r_*}^2 r_+ K\left(\Omega^2 - \Xi^2 l^2\right)\sqrt{\beta r_+ r_* \delta}}{2\Omega l^2 \Xi (r_+ + r_*)^3}. \tag{60}$$

The constant δ which appears in equations (57), (58), (59), and (60), is given by $\delta = 3\Omega^2{r_+}^2 - r_+ r_*(K\Xi^2 l^4 - 2\Omega^2) + \Omega^2{r_*}^2$. Moreover, we find two constraints for the parameters of the black hole solutions (14), such as

$$\frac{\Omega^2\Xi^2 l^4\left[-3{r_+}^2 + r_* r_+\left(Kl^2 - 2\right) - {r_*}^2\right]^2}{K^2\beta {r_+}^3 r_*(r_+ - r_*)^2(\Xi^2 l^2 - \Omega^2)^2} = \frac{l^4\Omega^2}{{r_+}^2 K(\Xi^2 l^2 - \Omega^2)^2(r_+ - r_*)^2\beta}, \tag{61}$$

$$\begin{aligned}&\frac{\Omega^2\Xi^2 l^4\left(Kl^2{r_*}^3 - {r_+}^3 - 2r_*{r_+}^2 - 3{r_*}^2 r_+\right)^2}{K^2\beta r_+{r_*}^5(r_+ - r_*)^2(\Omega^2 - \Xi^2 l^2)^2\left(Kl^4\Xi^2 r_* r_+ - 3\Omega^2{r_+}^2 - 2\Omega^2 r_* r_+ - \Omega^2{r_*}^2\right)} \\ &= \frac{l^4\Omega^2}{{r_+}^2 K(\Xi^2 l^2 - \Omega^2)^2(r_+ - r_*)^2\beta}.\end{aligned} \tag{62}$$

We note that equations (61) and (62) restrict the black hole parameters, in accord with existence of the real positive values for the outer horizon (and any other horizons), as the roots of the tri-quadratic algebraic equation

$$A(r) = 0, \tag{63}$$

where $A(r)$ is given in (15).

We note that both right and left temperatures (57) and (58) are positive definite [19]. We emphasize that $SL(2,R)_L \times SL(2,R)_R$ is a *local* hidden symmetry, for the solution space of massless scalar field in the near region of the rotating charged AdS black holes (14), in quadratic $f(T)$ gravity. The *local* hidden symmetry is generated by the vector fields (47)–(52), which are not periodic under the angular identification $\phi \sim \phi + 2\pi$. These symmetries can't be used to generate new global solutions from the old ones. This can be interpreted as a statement that the $SL(2,R)_L \times SL(2,R)_R$ symmetry is spontaneously broken down to $U(1)_L \times U(1)_R$ subgroup under the angular identification $\phi \sim \phi + 2\pi$. As a result, we can identify the left and right temperatures of the dual CFT. We recall the Cardy entropy formula for the dual 2D CFT with temperatures T_L and T_R

$$S_{\text{CFT}} = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right), \tag{64}$$

where c_L are c_R are the corresponding central charges for the left and right sectors. The central charges can be derived from the *asymptotic symmetry group* of the near-horizon (near-)extremal black hole geometry. There is no derivation for the central charges of the CFT dual to the non-extremal black holes, that we consider in this article. Of course, we expect that the conformal symmetry of the extremal black holes connects smoothly to those of the non-extremal black holes, for which the central charges are the same. The near-horizon extremal geometry for the spacetime (14) is still unknown, and it is not a straightforward task to find that, due to the tri-quadratic behaviour of the metric function $A(r)$. As a result, we turn the logic around and consider the favourite holographic situation, in which, the Cardy entropy (64) produces exactly the macroscopic entropy (25). Substituting equations (26), (57), and (58) into equation (64), we find the central charges

$$\begin{aligned} c &\equiv c_L = c_R \\ &= \frac{12\Xi\delta L\varpi(r_+ + r_*)^3}{lKr_+{}^2\left(\Xi^2 l^2 - \Omega^2\right)\left(r_+{}^3 + 2r_+{}^2 r_* + 3r_+ r_*{}^2 - l^2 K r_*{}^3\right)\left(Q\sqrt{6|\alpha|} + r_+{}^2\right)^2\sqrt{\beta r_+ r_* \delta}}, \end{aligned} \tag{65}$$

where $\varpi = r_+{}^2(r_+{}^2 Q\sqrt{6|\alpha|}/2 + 7r_+{}^4/18 + M|\alpha|r_+ - 3Q^2|\alpha|) - 7\sqrt{6|\alpha|}^3 Q^3/3$. We note that we only consider CFTs, in which the left and right central charges are equal, $c \equiv c_L = c_R$ [20, 21].

5. MICROSCOPIC ENTROPY IN DUAL CFT

We recall that asymptotic symmetry group of a spacetime is the group of allowed symmetries that obey the boundary conditions. As a result, the definition of the charge associated with a symmetry depends both on the action as well as boundary conditions. Hence, to compute the charges associated with asymptotic symmetry group of the black hole solution (14) with Maxwell potential $A_\mu = \tilde{\Phi}_\mu$ (19), we should consider all possible contributions from all different fields in the action (10). Asymptotic symmetries of the action (10) include diffeomorphisms ξ such that

$$\delta_\xi A_\mu = \mathcal{L}_\xi A_\mu, \tag{66}$$

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \tag{67}$$

as well as the following gauge transformation Λ for A_μ,

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda. \tag{68}$$

In equations (66) and (67), the Lie derivatives of the gauge field and the metric are given by

$$\mathcal{L}_\xi A_\mu = \xi^\nu F_{\mu\nu} + \nabla_\mu\left(A_\nu \xi^\nu\right), \tag{69}$$

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{70}$$

Hence, there are two contributions to the associated charge of asymptotic symmetry group of black hole solution (14). The contributions come from gravitational tensor and the $U(1)$ gauge field. So we have

$$Q_{\zeta,\Lambda} = \frac{1}{8\pi}\int_{\partial\Sigma}\left(k^g_\zeta[h;g] + k^A_{\zeta,\Lambda}[h,a;g,A]\right), \tag{71}$$

where h and a mean the infinitesimal variations of g and A fields, respectively. $\partial\Sigma$ is the boundary of a spatial slice. The gravitational contribution two-form $k^g_\zeta[h;g]$ is given by [22, 23]

$$k^g_\zeta[h;g] = -\delta\mathbf{Q}^g_\zeta + \mathbf{Q}^g_{\delta\zeta} + i_\zeta\mathbf{\Theta}[h] - \mathbf{E}_{\mathcal{L}}\left[\mathcal{L}_\zeta g, h\right], \tag{72}$$

where $\mathbf{\Theta}_\Phi = *(\phi d\Phi)$, $\mathbf{\Theta}[h] = \star\{(D^\beta h_{\alpha\beta} - g^{\mu\nu} D_\alpha h_{\mu\nu})dx^\alpha\}$ and

$$\mathbf{E}_{\mathcal{L}}\left[\mathcal{L}_\zeta g, h\right] = \star\left\{\frac{1}{2}h_{\alpha\gamma}\left(D^\gamma \zeta_\beta + D_\beta \zeta^\alpha\right)dx^\alpha \wedge dx^\beta\right\}, \tag{73}$$

and $\mathbf{Q}^g_\zeta$ is the Koumar two-form

$$\mathbf{Q}^g_\zeta = \frac{1}{2}\star\left(D_\mu \xi_\nu - D_\nu \xi_\mu\right)dx^\mu \wedge dx^\nu. \tag{74}$$

The last term in equation (71) is the contribution of one-form gauge field A, to the charge. In general for a $\hat{p}$-form P with the associated $(\hat{p}+1)$-form field strength R, the contribution is given by [24]

$$k^P_{\zeta,\Pi}[h,p;g,P] = -\delta\mathbf{Q}^P_{\zeta,\Pi} + \mathbf{Q}^P_{\delta\zeta,\delta\Pi} - i_\zeta \mathbf{\Theta}^{\mathbf{P}} - \mathbf{E}^P_{\mathcal{L}}\left[\mathcal{L}_\zeta P + d\Pi, p\right], \tag{75}$$

where

$$\mathbf{\Theta}^{\mathbf{P}} = p \wedge \star R, \tag{76}$$

$$\mathbf{E}^P_{\mathcal{L}}\left[\mathcal{L}_\zeta P + d\Pi, p\right] = \star\left\{\frac{1}{2(\hat{p}-1)!}p_{\mu\rho_1\cdots\rho_{\hat{p}-1}}\left(\mathcal{L}_\zeta P + d\Pi\right)_\nu^{\rho_1\cdots\rho_{\hat{p}-1}}dx^\mu \wedge dx^\nu\right\}, \tag{77}$$

and the two-form $\mathbf{Q}^P_{\zeta,\Pi}$ is

$$\mathbf{Q}^P_{\zeta,\Pi} = \left(i_\zeta P + \Pi\right)\wedge \star R. \tag{78}$$

The explicit expressions for the contributions to the charge (71) from gravity is given by

$$k^g_\zeta[h;g] = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\left\{\zeta^\sigma \nabla^\rho h - \zeta^\sigma \nabla_\lambda h^{\rho\lambda} + \zeta_\lambda \nabla^\sigma h^{\rho\lambda} + \frac{1}{2}h\nabla^\sigma \zeta^\rho - h^{\sigma\lambda}\nabla_\lambda \zeta^\rho + \frac{1}{2}h^{\lambda\sigma}\left(\nabla^\rho \zeta_\lambda + \nabla_\lambda \zeta^\rho\right)\right\}dx^\mu \wedge dx^\nu. \tag{79}$$

The Maxwell contribution is given by

$$\begin{aligned} k^A_{\zeta,\Lambda}[h,a;g,A] = &\frac{1}{8\pi}\epsilon_{\alpha\beta\mu\nu}\left\{\left(-\frac{1}{2}hF^{\mu\nu} + 2F^{\mu\rho}h^\nu_\rho - \delta F^{\mu\nu}\right)\left(\zeta^\sigma A_\sigma + \Lambda\right) - F^{\mu\nu}\zeta^\sigma a_\sigma - 2F^{\sigma\mu}\zeta^\nu a_\sigma\right\}dx^\alpha \wedge dx^\beta \\ &- \frac{1}{8}\epsilon^{\mu\nu}_{\alpha\beta}a_\mu\left(\mathcal{L}_\zeta A_\nu + \partial_\nu \Lambda\right)dx^\alpha \wedge dx^\beta. \end{aligned} \tag{80}$$

We should note that the last two terms in (79) as well as in (80) vanish for an exact Killing vector and an exact symmetry, respectively. We choose the proper boundary condition for the near-horizon metric as the same as one in [25]. Moreover, we choose the boundary conditions for the $U(1)$ gauge field

$$a_\mu \sim \mathcal{O}\left(r, 1/r^2, 1, 1/r\right), \tag{81}$$

to make sure that the conserved charges of the theory remain finite. We can show that the near-horizon metric has a class of commuting diffeomorphisms, labeled by $n = 0, \pm 1, \pm 2, \ldots$

$$\zeta_n = -e^{-in\phi}\left(\partial_\phi + inr\partial_r\right). \tag{82}$$

This diffeomorphism generates a Virasoro algebra without any central charge

$$[\zeta_m, \zeta_n] = -i(m-n)\zeta_{m+n}. \tag{83}$$

The charge (71) generates the symmetry $(\zeta, \Lambda)_n$ and the algebra of the asymptotic symmetric group is given by the Dirac bracket algebra of these charges

$$\begin{aligned} \left\{Q_{\zeta,\Lambda}, Q_{\tilde{\zeta},\tilde{\Lambda}}\right\}_{\text{D.B.}} &= \left(\delta_{\tilde{\zeta}} + \delta_{\tilde{\Lambda}}\right)Q_{\zeta,\Lambda} \\ &= \frac{1}{8\pi}\int_{\partial\Sigma}\left(k^g_\zeta\left[\mathcal{L}_{\tilde{\zeta}}g; g\right] + k^A_{\zeta,\Lambda}\left[\mathcal{L}_{\tilde{\zeta}}g, \mathcal{L}_{\tilde{\zeta}}A + d\tilde{\Lambda}; g, A\right]\right). \end{aligned} \tag{84}$$

Taking the background geometry $\hat{g}$ and field $\hat{A}$ as the near-horizon geometry and the near-horizon Maxwell field, we obtain

$$\left\{Q_{\zeta,\Lambda}, Q_{\tilde{\zeta},\tilde{\Lambda}}\right\}_{\text{D.B.}} = Q_{[(\zeta,\Lambda),(\tilde{\zeta},\tilde{\Lambda})]} + \frac{1}{8\pi}\int_{\partial\Sigma}\left(k^{\hat{g}}_\zeta\left[\mathcal{L}_{\tilde{\zeta}}\hat{g}; \hat{g}\right] + k^A_{\zeta,\Lambda}\left[\mathcal{L}_{\tilde{\zeta}}\hat{g}, \mathcal{L}_{\tilde{\zeta}}\hat{A} + d\tilde{\Lambda}; \hat{g}, \hat{A}\right]\right). \tag{85}$$

A straightforward and lengthy calculation shows that the algebra of the asymptotic symmetry group is a Viraso algebra generated by $(\zeta, \Lambda)_n$ with the central charge

$$c = c_g + c_A. \tag{86}$$

The two contributions to the central charge are generated by the last two central terms in (85), respectively. Moreover, we find that the chosen boundary conditions for the metric tensor and the gauge field keep all the conserved charges as well as the central charges completely finite. We find

$$\int_{\partial\Sigma} k^A_{\tilde{\zeta}_m,\Lambda_m}\left[\mathcal{L}_{\tilde{\zeta}_n}\hat{g}, \mathcal{L}_{\tilde{\zeta}_n}\hat{A} + d\tilde{\Lambda}; \hat{g}, \hat{A}\right] = 0, \tag{87}$$

which yield

$$c_A = 0. \tag{88}$$

This result explicitly shows that the non-gravitational fields do not contribute to the central charge of the dual CFT. Replacing the Dirac brackets by commutators yields a quantum Virasoro algebra with the central charge

$$c = c_g, \tag{89}$$

for the dual CFT corresponding to the black hole solution (14). To find the entropy of dual CFT, we need to find Frolov-Thorne temperature [26]. A straightforward calculation shows

$$T_L = T_R = \frac{1}{2\pi}. \tag{90}$$

Finally, we obtain the microscopic entropy for the dual CFT by using the Cardy relation, as

$$S = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right). \tag{91}$$

This microscopic result for the entropy is exactly the same as macroscopic entropy of black hole (14).

6. HOLOGRAPHY FOR CHARGED ROTATING $f(T)$ BLACK HOLES

In order to explore the hidden conformal symmetry for the charged rotating black hole solution (14), we consider a massless charged scalar field Φ with charge q, as a probe in the background of the black hole (14). The non-minimally coupled Klein-Gordon equation for the massless charged scalar field Φ is

$$\left(\nabla_\mu - iqA_\mu\right)\left(\nabla^\mu - iqA^\mu\right)\Phi = 0, \tag{92}$$

where A_μ is given by the same components as before in Section 2. As we are working in the $f(T)$ frame, the covariant derivative ∇_μ in the Klein-Gordon equation (92) where applied on an arbitrary vector v^ν, is given by the following equation,

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\rho\mu} v^\rho, \tag{93}$$

The connection $\Gamma^\nu{}_{\rho\mu}$ is constructed as [27]

$$\Gamma^\nu{}_{\rho\mu} = \mathring{\Gamma}^\nu{}_{\rho\mu} + K^\nu{}_{\rho\mu}, \tag{94}$$

where $\mathring{\Gamma}^\nu{}_{\rho\mu}$ is the zero-torsion Christoffel (Levi-Civita) connection, and $K^\nu{}_{\rho\mu}$ is the contortion tensor. The contorsion tensor is defined in terms of the torsion tensor by

$$K^\nu{}_{\rho\mu} = \frac{1}{2}\left(T_\mu{}^\nu{}_\rho + T_\rho{}^\nu{}_\mu - T^\nu{}_{\rho\mu}\right). \tag{95}$$

Since the black hole solution (14) has three Killing vectors, we separate the coordinates in the scalar field as

$$\Phi(t, r, \phi, z) = \exp(-i\omega t + im\phi + ikz)R(r). \tag{96}$$

Plugging equation (96) in the Klein-Gordon equation (92), we find that the radial function $R(r)$ satisfies

$$B(r)\frac{d^2R(r)}{dr^2} + \frac{6}{rA(r)}\left(1/4rB(r)\frac{dA(r)}{dr} + 1/12rA(r)\frac{dB(r)}{dr} + A(r)B(r)\right)\frac{dR}{dr} + V(r)R(r) = 0, \tag{97}$$

where $V(r)$ is given by

$$V(r) = V_0(r) + qV_1(r) + q^2 V_2(r). \tag{98}$$

The different terms in (98) are

$$V_0(r) = \frac{-\left(k^2 l^4 \Xi^4 + \left(\left(-2\Xi^2 k^2 + \omega^2\right)\Omega^2 - 2\omega m \Xi \Omega + m^2 \Xi^2\right) l^2 + k^2 \Omega^4\right) l^2 A(r)}{r^2 \left(\Xi^2 l^2 - \Omega^2\right)^2 A(r)} + \frac{\left(\Xi l^2 \omega - \Omega m\right)^2}{A(r)\left(\Xi^2 l^2 - \Omega^2\right)^2}, \tag{99}$$

$$V_1(r) = \frac{2\left(-\Xi l^2 \omega + \Omega m\right)\left(Q\sqrt{6|\alpha|} + 3r^2\right) Q}{3r^3 A(r)\left(-\Xi^2 l^2 + \Omega^2\right)}, \tag{100}$$

$$V_2(r) = \frac{\left(2\sqrt{6|\alpha|} Q r^2 + 3r^4 + 2Q^2|\alpha|\right) Q^2}{3r^6 A(r)}. \tag{101}$$

We obtain the quadratic Casimir operator, which obeys the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ algebra, from any of two sets of these operators as

$$\begin{aligned} \mathcal{H}^2 = \overline{\mathcal{H}}^2 &= -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1) \\ &= \frac{1}{4}\left(y^2 \partial_y^2 - y\partial_y\right) + y^2 \partial_+ \partial_-. \end{aligned} \tag{102}$$

By writing the Casimir operator in terms of the coordinates (t, r, ϕ)

$$\begin{aligned} \mathcal{H}^2 = {} & (r - r_+)(r - r_*)\partial_r^2 + (2r - r_+ - r_*)\partial_r + \frac{r_+ - r_*}{r - r_*}\left(\frac{n_L - n_R}{4\pi G}\partial_\phi - \frac{T_L - T_R}{4G}\partial_t\right)^2 \\ & - \frac{r_+ - r_*}{r - r_+}\left(\frac{n_L + n_R}{4\pi G}\partial_\phi - \frac{T_L + T_R}{4G}\partial_t\right)^2, \end{aligned} \tag{103}$$

we notice the relation between the radial equation (97) and the eigenvalue-eigenfunction equation for the quadratic Casimir operator (103), where $G = n_L T_R - n_R T_L$.

First, by considering $q = 0$ for the scalar probe in the simplified radial equation, we find the correspondent CFT of the black hole in the J picture. By comparing the Casimir operator (103) and the simplified radial equation, we rewrite the simplified radial equation in terms of the $SL(2,\mathbb{R})$ quadratic Casimir operator, as

$$\mathcal{H}^2 R(r) = \overline{\mathcal{H}}^2 R(r) = -\mathcal{C} R(r), \tag{104}$$

and we find

$$n_L^J = \frac{f+1}{4(g-h)} \frac{r_+^3 K (r_+ - r_*)\left(\Xi^2 l^2 - \Omega^2\right)}{\left(Q\sqrt{6|\alpha|} + r_+^2\right) l^2} \sqrt{\frac{1}{3\Xi^2 r_-^2 - r_*\left(K\Omega^2 - 2\Xi^2\right) r_+ + \Xi^2 r_*^2}}, \tag{105}$$

$$n_R^J = \frac{f-1}{4(g-h)} \frac{r_+^3 K (r_+ - r_*)\left(\Xi^2 l^2 - \Omega^2\right)}{\left(Q\sqrt{6|\alpha|} + r_+^2\right) l^2} \sqrt{\frac{1}{3\Xi^2 r_-^2 - r_*\left(K\Omega^2 - 2\Xi^2\right) r_+ + \Xi^2 r_*^2}}, \tag{106}$$

$$T_L^J = \frac{1}{4\pi(g-h)} \frac{r_+^3 K (r_+ - r_*)\left(\Xi^2 l^2 - \Omega^2\right)}{\left(Q\sqrt{6|\alpha|} + r_+^2\right)} \frac{\sqrt{\frac{\left(K\Omega^2 r_*^3 + \left(-r_+^3 - 2r_* r_+^2 - 3r_*^2 r_+\right)\Xi^2\right) r_+}{\left(\left(-3r_+^2 - 2r_+ r_* - r_*^2\right)\Xi^2 + K\Omega^2 r_+ r_*\right) r_*^2}} + 1}{\sqrt{3\Omega^2 r_+^2 - r_*\left(K\Xi^2 l^4 - 2\Omega^2\right) r_+ + \Omega^2 r_*^2}}, \tag{107}$$

$$T_R^J = \frac{1}{4\pi(g-h)} \frac{r_+^3 K (r_+ - r_*)\left(\Xi^2 l^2 - \Omega^2\right)}{\left(Q\sqrt{6|\alpha|} + r_+^2\right)} \frac{\sqrt{\frac{\left(K\Omega^2 r_*^3 + \left(-r_+^3 - 2r_* r_+^2 - 3r_*^2 r_+\right)\Xi^2\right) r_+}{\left(\left(-3r_+^2 - 2r_+ r_* - r_*^2\right)\Xi^2 + K\Omega^2 r_+ r_*\right) r_*^2}} - 1}{\sqrt{3\Omega^2 r_+^2 - r_*\left(K\Xi^2 l^4 - 2\Omega^2\right) r_+ + \Omega^2 r_*^2}}. \tag{108}$$

In equations (105)–(108), the quantities f, g, and h are given explicitly in [28].

Second, by considering only the zero-mode of the angular momentum $m = 0$ for the scalar probe in the simplified radial equation, and the scalar probe expansion, we find the second holographic description which is Q picture. Based on the Casimir operator, the simplified radial equation can be written as

$$\mathcal{H}^2 R(r) = \tilde{\mathcal{H}}^2 R(r) = -\mathcal{C} R(r), \tag{109}$$

and we find the CFT parameters as

$$n_R^Q = -\frac{S-1}{4UV}, \tag{110}$$

$$n_L^Q = -\frac{S+1}{4UV}, \tag{111}$$

$$T_R^Q = \frac{\sqrt{6}(Y-1)}{-4\pi XV}, \tag{112}$$

$$T_L^Q = \frac{\sqrt{6}(Y+1)}{-4\pi XV}, \tag{113}$$

where the quantities S, U, V, X, and Y are given explicitly in [28].

7. CONCLUSIONS

In this presentation which is based entirely on the published paper [4, 28, 29], we extend the concept of black hole holography to the non-extremal 4D rotating charged AdS black holes in $f(T)$-Maxwell theory with a negative cosmological constant. We explicitly construct the hidden conformal symmetry for the rotating black holes in $f(T)$-Maxwell theory with a negative cosmological constant. We mainly consider the near-horizon region, as the metric function which determines the event horizon, is a triple-quadratic equation. In this region, we show that the radial equation of the scalar wave function could be written as the $SL(2,R)_L \times SL(2,R)_R$ squared Casimir equation, indicating a *local* hidden conformal symmetry acting on the solution space. The conformal symmetry is spontaneously broken under the angular identification $\phi \sim \phi + 2\pi$, which suggests the rotating charged AdS black holes in quadratic $f(T)$ gravity, should be dual to the finite temperatures T_L and T_R mixed state, in the 2D CFT. Instead of calculating the central charges using the *asymptotic symmetry group*, we calculated the central charges by assuming the Cardy entropy for the dual CFT, matches the macroscopic Bekenstein-Hawking entropy. These results suggest that rotating charged AdS black holes in quadratic $f(T)$ gravity with particular values of M, Ω, Q, and $|\alpha|$, are dual to a 2D CFT.

It is an open question to find the near-horizon (near-)extremal geometry of the rotating charged AdS spacetime in quadratic $f(T)$ gravity. We may calculate the central charges using the *asymptotic symmetry group* to confirm our results in this article. We can also study on various kinds of superradiant scattering off the near-extremal black hole as a further evidence to support the holographic picture for the non-extremal 4D rotating charged AdS black holes in $f(T)$-Maxwell theory with a negative cosmological constant. We leave addressing these open questions for future articles.

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