

Flavor and \mathcal{CP} from String Theory

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Abstract

Modular transformations of string theory are shown to play a crucial role in the discussion of discrete flavor symmetries in the Standard Model. They include \mathcal{CP} transformations and provide a unification of \mathcal{CP} with traditional flavor symmetries within the framework of the “eclectic flavor” scheme. The unified flavor group is non-universal in moduli space and exhibits the phenomenon of “Local Flavor Unification”, where different sectors of the theory (like quarks and leptons) can be subject to different flavor structures.

Keywords: flavor symmetry, string theory constructions, modular transformations, eclectic flavor group, local flavor unification
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1. OUTLINE

We shall discuss the world of flavor from a top-down point of view. The topics include

- Traditional flavor symmetries
- Modular flavor symmetries
- Natural appearance of a \mathcal{CP} symmetry
- The concept of eclectic flavor symmetry
- A unified picture of quark- and lepton-flavor: flavor groups localized in moduli space
- Spontaneous breakdown of flavor and \mathcal{CP} symmetry
- Comparison of top-down and bottom-up constructions
- Conclusion and outlook

In this talk (at BSM2021, Zewail City, Egypt by HPN) we are reporting about joint work in various combinations with Alexander Baur, Moritz Kade and Andreas Trautner.

2. DISCRETE FLAVOR SYMMETRIES

2.1. Bottom-up

Most of the parameters of the $SU(3) \times SU(2) \times U(1)$ Standard Model of particle physics concerns the flavor sector: masses and mixing angles of quarks and leptons. Their origin is not yet understood. There are many fits to the data from a bottom-up perspective (see e.g. [1]) that postulate discrete flavor symmetries (like e.g. $S_3, A_4, S_4, A_5, \Delta(27), \Delta(54)$, etc. [2]) and choose specific representations of these groups for the Standard Model fermions. The data seem to require different models for quark- and lepton-sector with qualitative different flavor structures. While in the quark sector we observe small mixing angles, this is different in the lepton sector. Moreover, flavor symmetries hold only approximately and have to be broken spontaneously. This requires the introduction of so-called flavon fields whose vacuum expectation values are responsible for this breakdown. With the choice of these flavon fields one introduces many additional parameters. With specific choices of these parameters (and the discrete flavor group as well as the relevant representations) the model building from the bottom-up perspective leads to many different successful fits to masses and mixing angles of matter fields. Still there is not yet a clear preference for a given class of models and we seem to need a top-down explanation of the flavor puzzle.

2.2. Top-down

Such a top-down explanation might come from string theory. Discrete symmetries have their origin in the symmetries of compact extra dimensions as well as specific string theory selection rules (that arise from conformal symmetry on the string worldsheet) [3]. As an illustration, we consider two-dimensional orbifold compactifications of heterotic string theory [4, 5, 6]. They are directly relevant for flavor symmetries of six-dimensional string compactifications with an elliptic fibration. They provide the chiral spectrum of quarks and leptons at low energies within the $SU(3) \times SU(2) \times U(1)$ Standard Model [7, 8, 9]. In addition, they display abundant discrete symmetries for flavor physics that might allow a connection to the existing bottom-up constructions [3, 10, 11].

As we shall see, the string theory picture predicts the existence of

- traditional flavor symmetries that are universal in moduli space,
- modular flavor symmetries which are non-universal in moduli space,
- a natural candidate for a \mathcal{CP} symmetry.

The non-universality of modular flavor symmetries in moduli space leads to the phenomenon of ‘‘Local Flavor Unification’’ at specific points (or higher-dimensional sub-regions) in moduli space [11]. This might allow the explanation of the different flavor structures in the quark- and lepton-sector of the Standard Model. The spontaneous breakdown of flavor and \mathcal{CP} symmetries can be understood as a motion in moduli space (away from these specific points or subregions).

3. FLAVOR SYMMETRY FROM STRING THEORY

Discrete symmetries can arise from geometry and string selection rules. As an illustrative example, let us consider the two-dimensional \mathbb{Z}_3 orbifold $\mathbb{T}^2/\mathbb{Z}_3$ (see Figure 1). The lattice vectors e_1 and e_2 have the same length and are separated by an angle of 120 degrees (to allow for the \mathbb{Z}_3 twist). The spectrum of the string theory contains untwisted modes, winding modes as well as twisted modes. The latter are located at the fixed points X, Y, Z of the orbifold twist. In this picture, we obtain an S_3 symmetry from the interchange of the fixed points. Orbifold selection rules (from the underlying conformal field theory on the string worldsheet) add a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. The multiplicative closure of S_3 and $\mathbb{Z}_3 \times \mathbb{Z}_3$ leads to a group with 54 elements called $\Delta(54)$, a discrete non-Abelian subgroup of $SU(3)$, and could e.g. describe the flavor symmetry of three families of quarks [12]. The twisted fields transform as a 3-dimensional representation of $\Delta(54)$.

We see that even such simple systems lead to sizeable flavor groups. A full determination of the flavor symmetries in six compact dimensions could then lead to a complicated flavor structure that requires a general method to classify these flavor symmetries. Such a mechanism has been identified via the outer automorphisms [11, 13] of the so-called Narain space group [14, 15, 16, 17]. Here, we shall not be able to give a full derivation of this fact but illustrate it in our examples (for a $D = 2$ -dimensional torus and for an orbifold).

On a string there are D right-moving and D left-moving degrees of freedom: $Y = (y_R, y_L)$. A geometrical compactification of strings on a D -dimensional torus amounts to a compactification of Y on a $2D$ -dimensional auxiliary torus, obtained by demanding the boundary condition

$$Y = \begin{pmatrix} y_R \\ y_L \end{pmatrix} \sim Y + E \hat{N} = \begin{pmatrix} y_R \\ y_L \end{pmatrix} + E \begin{pmatrix} n \\ m \end{pmatrix}, \quad n, m \in \mathbb{Z}^D,$$

that defines a so-called $2D$ Narain lattice (four-dimensional for the $D = 2$ -torus). It includes the string’s winding and Kaluza–Klein quantum numbers n and m , respectively. The Narain vielbein matrix E depends on the moduli of the torus: radii, angles and antisymmetric tensor fields. It is important to note that here we do include not only the (momentum) Kaluza–Klein modes of the

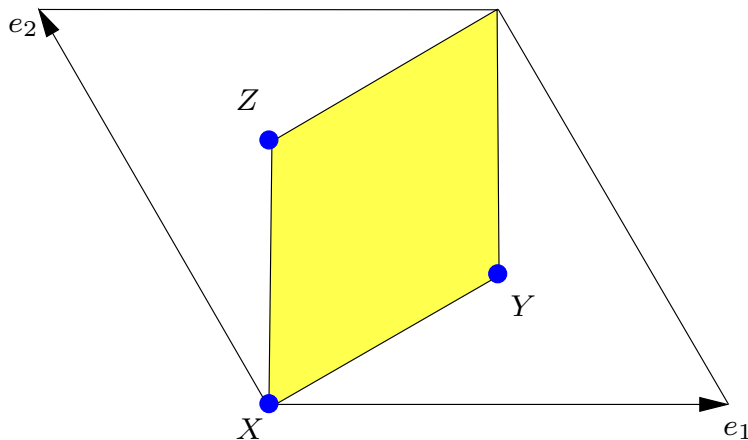


FIGURE 1: \mathbb{Z}_3 orbifold (yellow-shaded region) with the three fixed points X, Y, Z . Twisted strings are located at these fixed points.

compactified space but also the string winding modes on the nontrivial cycles of the torus. In order to arrive at a D -dimensional orbifold in the Narain formulation, we now introduce a \mathbb{Z}_K orbifold twist Θ that leads to the identification

$$Y \sim \Theta^k Y + E \hat{N}, \quad \text{where } \Theta = \begin{pmatrix} \theta_R & 0 \\ 0 & \theta_L \end{pmatrix} \quad \text{and } \Theta^K = \mathbb{1}_{2D},$$

with θ_L, θ_R elements of $SO(D)$ and $k = 0, \dots, K-1$. For a symmetric orbifold, we set $\theta_L = \theta_R$. The Narain space group, built by elements $g = (\Theta^k, E \hat{N})$, is then generated by

$$\text{the twist } (\Theta, 0) \text{ and shifts } (1, E_i) \text{ for } i = 1, \dots, 2D,$$

where we do not include roto-translations (that would correspond to off-diagonal terms in Θ). Flavor symmetries correspond to the outer automorphisms of this Narain space group. Outer automorphisms map the group to itself but are not elements of the group. Note that they also include modular transformations that interchange winding and momentum modes.

4. MODULAR TRANSFORMATIONS IN STRING THEORY

Duality symmetries are frequently found in string theory. One prominent example among them is T-duality that exchanges winding and momentum modes. Consider strings on a circle of radius R . The discrete spectrum of the momentum (Kaluza–Klein) modes is governed by $1/R$, while for the winding modes we find a spacing R . Winding states become heavy when the radius of the circle increases. The T-duality transformation of string theory exchanges

$$\text{winding} \mapsto \text{momentum}$$

modes and, simultaneously

$$R \mapsto \alpha'/R.$$

This maps a theory to its T-dual with a self-dual point at

$$R^2 = \alpha' = 1/M_{\text{string}}^2,$$

where M_{string}^2 is the string tension. If the string scale is large, then the low energy effective field theory describes the momentum modes and the winding states will be heavy. This raises the question whether T-duality can be relevant to flavor physics. Before we answer this question to the positive, let us generalize the circle compactification to higher-dimensional tori. This leads us to modular transformations that still exchange winding and momentum modes and act nontrivially on the moduli of the torus. In $D = 2$ these transformations are connected to the group $SL(2, \mathbb{Z})$ acting on the moduli (Kähler and complex structure modulus) of the $D = 2$ -torus. The group $SL(2, \mathbb{Z})$ is generated by two elements

$$S, T \quad \text{with } S^4 = \mathbb{1}, \quad S^2 T = T S^2 \quad \text{and } (ST)^3 = \mathbb{1}.$$

For each modular group $SL(2, \mathbb{Z})$, there exists an associated modulus M that transforms as

$$S: M \mapsto -\frac{1}{M} \quad \text{and} \quad T: M \mapsto M + 1.$$

Further transformations include mirror symmetry (exchanging Kähler and complex structure modulus) as well as the transformation

$$M \mapsto -\bar{M},$$

where \bar{M} denotes the complex conjugate of M . The latter turns out to be a universal and natural candidate for a \mathcal{CP} symmetry. String dualities give important constraints on the action of the theory via the modular group $SL(2, \mathbb{Z})$. Combining S and T , the general transformation $\gamma \in SL(2, \mathbb{Z})$ of the modulus is given by

$$M \xrightarrow{\gamma} \frac{aM + b}{cM + d}$$

with $ad - bc = 1$ and integers a, b, c, d . The value of M (originally in the upper complex half plane) is then restricted to the fundamental domain, as shown in Figure 2. Matter fields ϕ transform similar to modular forms of weight k

$$\phi \xrightarrow{\gamma} (cM + d)^k \rho(\gamma) \phi \quad \text{for } \gamma \in SL(2, \mathbb{Z}),$$

where $(cM + d)^k$ is known as automorphy factor and $\rho(\gamma)$ is a representation of γ . It is important to realize that $(\rho(T))^N = \mathbb{1}$ even though $T^N \neq \mathbb{1}$ such that $\rho(\gamma)$ generates a finite group, the so-called finite modular group, as we will discuss shortly in more detail. Among others, the modular weights k of the fields are important ingredients for flavor model building. The Yukawa couplings are modular forms and transform properly (with the modular weights determined by the theory under consideration). Then, in a

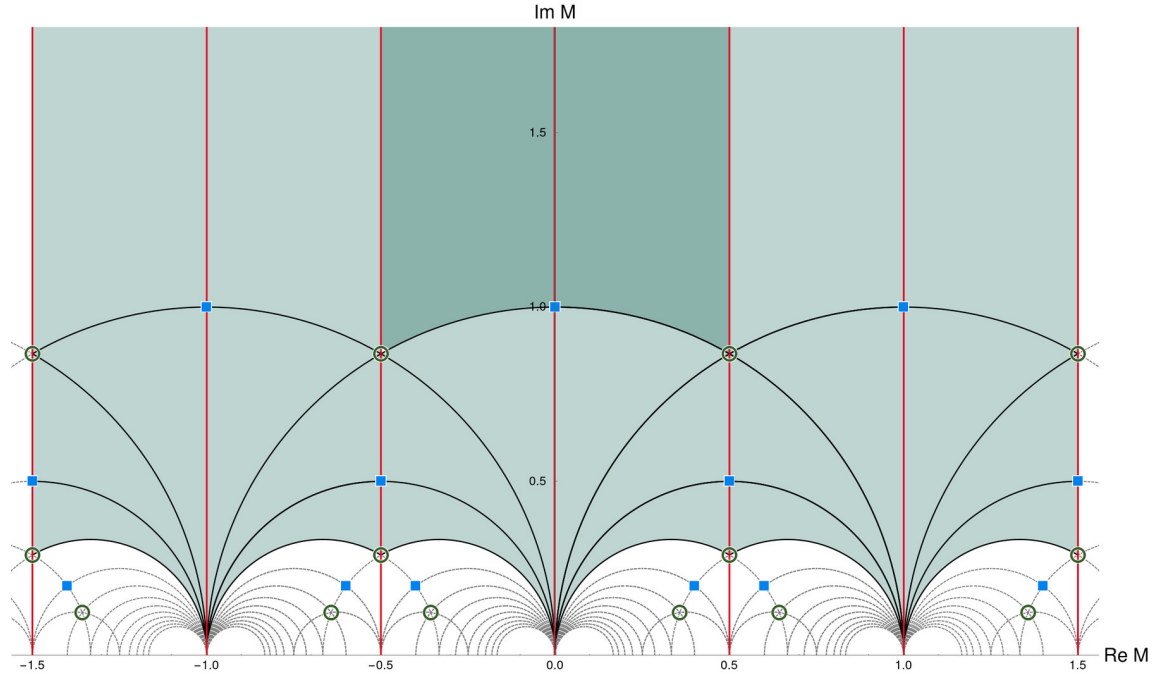


FIGURE 2: The dark shaded region is the fundamental domain of $SL(2, \mathbb{Z})$ (extending to $M = i\infty$). The light shaded region is the fundamental domain of the finite modular group T' .

supersymmetric theory, the combination $G = K + \log(\mathcal{W}\overline{\mathcal{W}})$ of Kähler potential K and superpotential \mathcal{W} must be invariant under modular transformations.

Let us now consider the potential relevance of modular symmetries for flavor physics. Again, we would like to illustrate this with the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold (see Figure 1). On this orbifold some of the moduli are frozen. Here, the length of the lattice vectors e_1 and e_2 are equal with an angle of 120 degrees between them. This gives restrictions on how modular transformations act on the matter fields. For twisted string modes, the coefficients a, b, c, d in the transformation

$$M \mapsto \frac{aM + b}{cM + d}$$

are defined only modulo 3, which amounts to $N = 3$ in $(\rho(T))^N = \mathbb{1}$. This constraint indicates that for $\mathbb{T}^2/\mathbb{Z}_3$ a modular transformation γ acts trivially on matter fields if γ belongs to the so-called principal congruence subgroup¹ $\Gamma(3) \subset SL(2, \mathbb{Z})$. For more general \mathbb{Z}_K orbifolds, other infinite subgroups of $SL(2, \mathbb{Z})$ arise. E.g. for $K = 2$, the modular symmetry of the \mathbb{Z}_2 orbifold is related to $\Gamma(2)$. The quotients $\Gamma_N = PSL(2, \mathbb{Z})/\Gamma(N)$ and their double covers $\Gamma'_N = SL(2, \mathbb{Z})/\Gamma(N)$ are called finite modular groups, and these are the groups that will become relevant as discrete flavor symmetries. In our example, we have $N = 3$ and the finite modular group is $\Gamma'_3 \cong T' \cong SL(2, 3)$ (the double cover of $\Gamma_3 \cong A_4$, the group of even permutations of four objects). The fundamental domain of $\Gamma'_3 \cong T'$ is bigger than that of $SL(2, \mathbb{Z})$, as shown in Figure 2. If we include the \mathcal{CP} transformation that descends from the transformation $M \mapsto -\overline{M}$, we obtain the full finite modular group $GL(2, 3)$, or [48, 29] according to the classification of GAP [18]. It is a group with 48 elements (as the first entry in the bracket indicates).

In a full string construction that allows for predictions at low energies, the modulus M must be stabilized. If the modulus M is fixed at a generic point in moduli space, the modular symmetry breaks down completely. However, if the modulus adopts its value at a special fixed point (or fixed higher-dimensional sub-locus) in moduli space, some modular generators are unbroken and build a residual modular symmetry. If for example M takes the value i , only the modular subgroup generated by S is preserved, see Figure 3.

5. THE ECLECTIC FLAVOR GROUP

We can now summarize the discrete flavor symmetries of string theory that can be determined via the outer automorphisms of the Narain space group. We have

- **traditional flavor symmetries** that are universal in moduli space. In our example this corresponds to the flavor group $\Delta(54)$.
- a subset of the **finite modular group** that acts as a symmetry at specific fixed points (or fixed higher-dimensional sub-loci) in moduli space.

¹Recall that $\Gamma(N)$ is defined abstractly as the infinite subgroup of $SL(2, \mathbb{Z})$, where $a, d = 1 \pmod N$ and $b, c = 0 \pmod N$.

- at these “points” we have an **enhanced flavor symmetry** that combines the traditional flavor symmetry with the unbroken elements of the modular symmetry group (in our example, the finite modular group $GL(2,3)$).
- in addition, matter fields have **specific modular weights** k , which leads to further restrictions on the action of the theory (for example, on the Yukawa couplings). Moreover, they give rise to further enhancements of the symmetries in form of R -symmetries.

The full flavor symmetry is thus non-universal in moduli space. At specific “points” we have an enhancement of the traditional flavor symmetry (which itself is universal in moduli space).

This brings us to the definition of the “Eclectic Flavor Group” [19]. Let us discuss this first in a simplified situation where we ignore the contributions of the automorphy factors that accompany the transformation of matter fields with a nontrivial weight. We then have to combine

- the traditional flavor group ($\Delta(54)$ in our example) and
- the finite modular flavor group that transforms the moduli as well (here T' , or $GL(2,3)$ when \mathcal{CP} is included).

The eclectic flavor group is now defined as the multiplicative closure of these groups. For the \mathbb{Z}_3 orbifold we obtain

- $\Omega(1) \cong [648, 533]$ from $\Delta(54)$ and T' ,
- $[1296, 2891]$ from $\Delta(54)$ and $GL(2,3)$, including \mathcal{CP} .

The eclectic flavor group is the largest possible flavor group for a given system, but it is not necessarily linearly realized. Part of it is spontaneously broken by the vacuum expectation values of the moduli. For generic values of the moduli only the traditional flavor symmetry remains unbroken.

So far our simplified discussion. As we had already stressed earlier, the restrictions of modular symmetry are more severe than just represented by the finite modular flavor group (here T'). In addition, we have to take into account the automorphy factors that come with the modular weights of the matter fields. These modular weights are determined in the specific string theory under consideration [20, 21] and lead to an extension of the eclectic flavor group via additional R -symmetries. This has been discussed in detail in refs. [22, 23]. For the \mathbb{Z}_3 orbifold, we can identify a symmetry \mathbb{Z}_9^R that extends $\Omega(1) \cong [648, 533]$ to $\Omega(2) \cong [1944, 3448]$ without \mathcal{CP} . If we further include \mathcal{CP} , we obtain a group with 3888 elements.

6. MODULAR FLAVOR: TOP-DOWN VERSUS BOTTOM-UP

6.1. Top-Down

We have seen that the flavor symmetries in the top-down (TD) approach are very restrictive and lead to a scheme with high predictive power. Some of the reasons are:

- restrictions from the finite modular flavor group Γ_N or its double cover Γ'_N ,
- specific representations for matter fields are selected and fixed by the underlying string symmetry,
- these restrictions also hold for the values of the modular weights and the resulting “shaping symmetries”,
- the role of flavon fields is partially played by the moduli, and
- the modular flavor symmetries are always accompanied by a traditional flavor symmetry (with further restrictions on superpotential and Kähler potential of the theory [24]).

We arrive at a large eclectic flavor group that reflects the symmetries of the underlying string theory and is not subject to arbitrary choices allowed in the bottom-up approach.

Let us illustrate these facts in the framework of our \mathbb{Z}_3 orbifold example, as discussed in ref. [24]. The twisted states at the three fixed points correspond to a triplet representation of $\Delta(54)$. Other states in the low energy sector correspond to singlets (both trivial and nontrivial). Winding states correspond to doublets of $\Delta(54)$ and are heavy [25]. The twisted states, however, do not transform as a triplet representation of the finite modular group T' , but as a combination of a nontrivial doublet $2'$ and the trivial singlet 1 . Other light states transform as singlets under T' . The modular weights of the twisted fields are restricted to the values $k = -2/3$ or $-5/3$ (in the first twisted sector) and cannot be chosen freely (matter fields from the second twisted sector have $k = -1/3$ and $2/3$). All of these ingredients, the traditional flavor symmetry, the modular flavor symmetry and the specific automorphy factors lead to a very restrictive eclectic scheme.

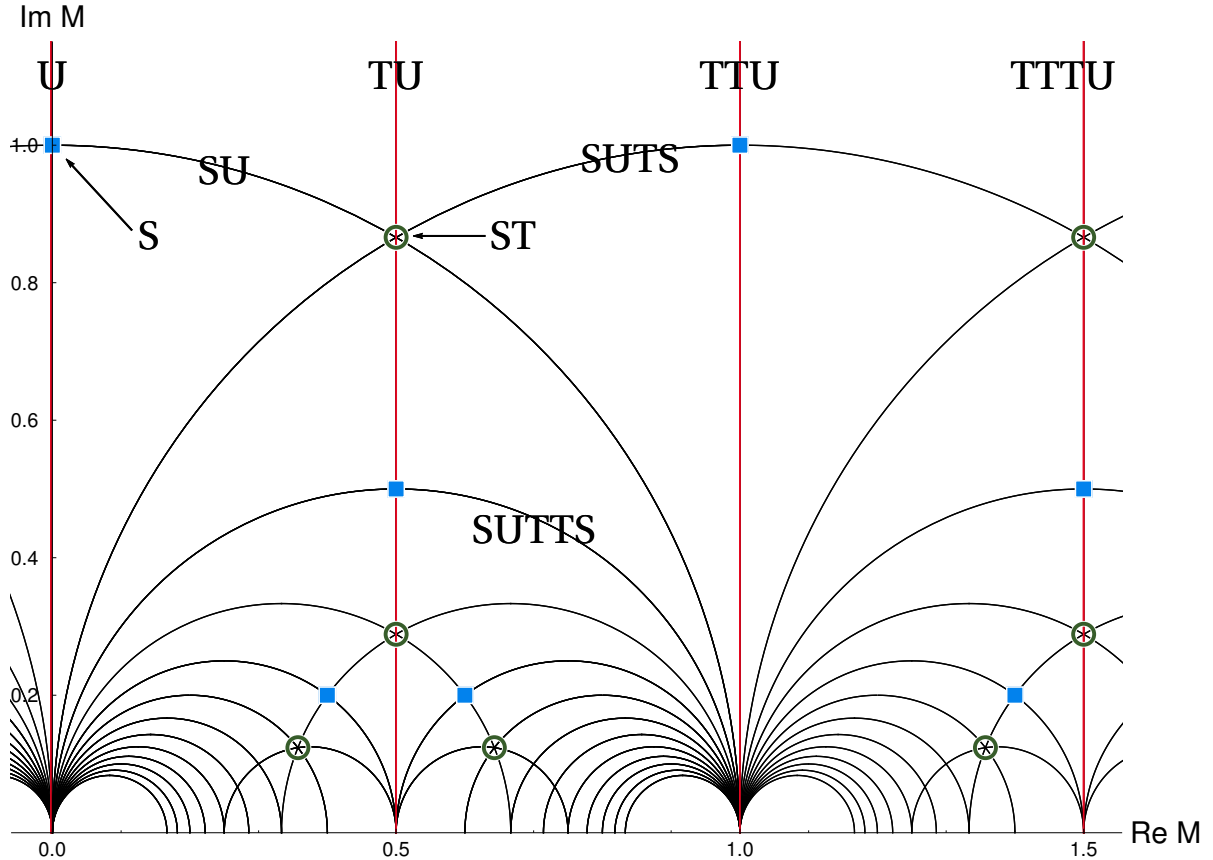


FIGURE 3: Unbroken modular symmetries at various fixed points and fixed lines in moduli space, where U denotes the \mathcal{CP} -like transformation $M \mapsto -\bar{M}$.

6.2. Bottom-Up

The application of modular symmetries for flavor physics has been pioneered in a remarkable paper of Feruglio [26]. In his example he considers the lepton sector with finite modular flavor group $\Gamma_3 \cong A_4$. He assigns the left-handed leptons to the triplet representation of A_4 and the right-handed leptons to a combination $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$ of trivial and nontrivial singlets. Specific choices of the modular weights of matter fields lead to a successful fit for lepton masses and mixing angles. By now there have been many more model constructions based on other finite modular groups like Γ_4 or Γ_5 and their double covers with and without \mathcal{CP} (see e.g. refs. [27, 28, 29, 30, 31, 32, 33]). Model building includes free choices of representations and modular weights of the matter fields. Further, admissible terms in the Kähler potential that may be phenomenologically relevant are typically disregarded [34]. In these constructions the presence of an additional traditional flavor symmetry is usually ignored, although it might further restrict the couplings of the theory and influence the results of the fit.

6.3. The Representation Dilemma

Given the status of the field, at the moment a comparison of top-down (TD) and bottom-up (BU) constructions appears to be pretty difficult. First, we have to match the groups. While BU-approaches use a wide variety of finite modular flavor groups, the TD-approaches seem to be very restrictive. For example, the group A_4 is difficult to find. Observe that it is not a subgroup of its double cover T' , which we find in the \mathbb{Z}_3 orbifold. From that point of view, it might be more rewarding to intensify model building with modular flavor group T' (for recent discussions, see ref. [29]).

The next step represents an even more serious challenge: the choice of the representations of the matter fields. Successful fits in the BU-approach typically assume a variety of representations of matter fields, like the triplets and various nontrivial singlets mentioned earlier. In the TD-approach, the choices seem to be much more limited. If we consider the model based on T' , we see that the candidate matter fields do not transform as a triplet but as $\mathbf{1} \oplus \mathbf{2}'$ and in addition we do not have all the necessary nontrivial singlets in the low energy spectrum. The same is true for the modular weights of the matter fields which are fixed in the TD-approach.

To match TD- and BU-approaches more work is needed. On the one hand we need more explicit TD-constructions to gain a better control of the relevant groups and representations. On the other hand one might try to consider BU-models with more limited number of representations and modular weights. One would also try to understand the role of the traditional flavor symmetry that comes automatically in TD constructions and that has widely been neglected in modular BU models. In addition, the spontaneous breakdown of the traditional flavor symmetry has to be studied in detail. Much work remains to be done.

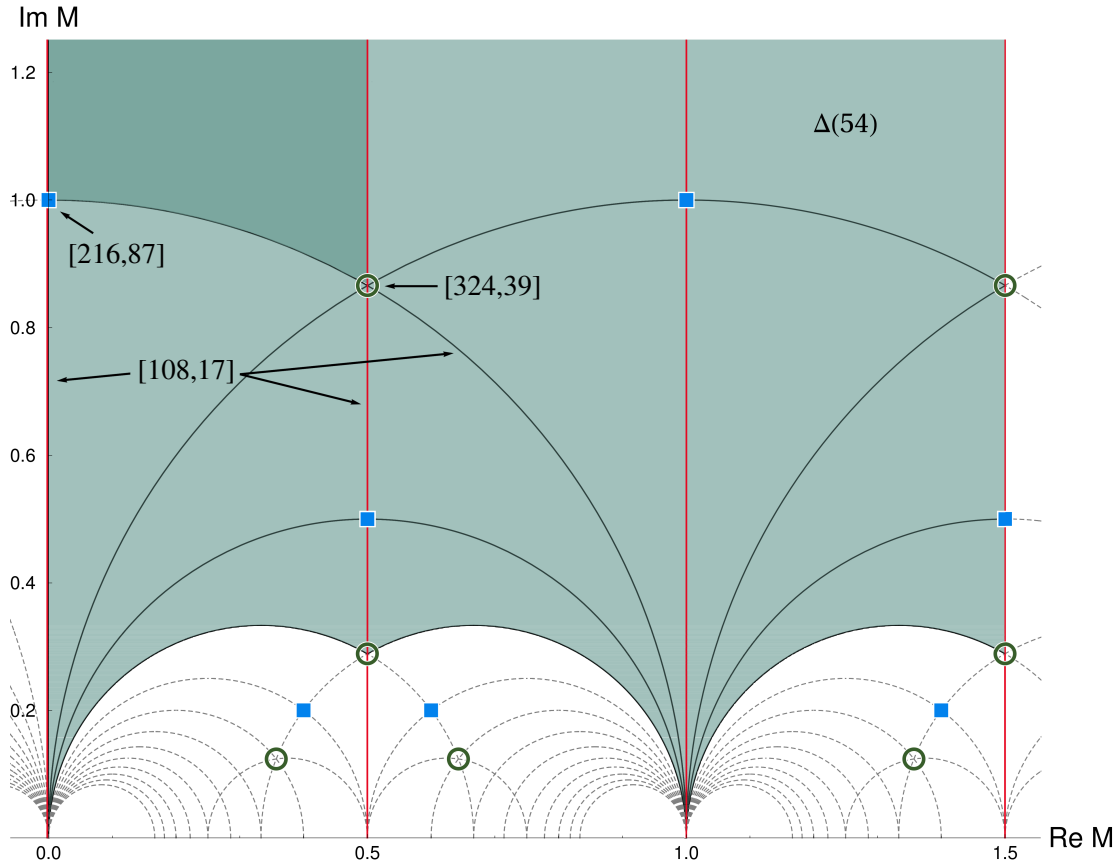


FIGURE 4: Points and curves of the enhancement of flavor symmetries. On the vertical red lines and the black semi-circles the traditional flavor group $\Delta(54)$ gets enhanced to $[108,17]$. Further enhancements appear when two or three of these lines intersect.

7. LOCAL FLAVOR UNIFICATION

The eclectic flavor group appears as the multiplicative closure of the traditional flavor group and the (finite) modular symmetries. The former holds universally in moduli space while the latter are broken for generic values of the moduli. Nonetheless, for some specific values of the moduli (fixed sub-loci of the modular transformations), part of the modular symmetries are unbroken. This results in an enhancement of the traditional flavor group, leading to so-called “Local Flavor Unification”. The full flavor symmetry is non-universal in moduli space and the spontaneous breakdown of modular flavor symmetry can be understood as a motion in moduli space. If we move away from these “fixed points”, part of the flavor symmetry is broken. In the following we shall illustrate this mechanism within two specific examples: the \mathbb{Z}_3 and the \mathbb{Z}_2 orbifold.

7.1. \mathbb{Z}_3 orbifold

Here, we have the finite modular group T' , being the double cover of $\Gamma_3 \cong A_4$. The fundamental domain of its moduli space is shown in Figure 2. It reflects the restriction of the modular transformations S and T (with the $\text{mod } 3$ condition) on the modulus M . In addition, we have the \mathcal{CP} -like transformation

$$U : M \mapsto -\bar{M},$$

which reduces this fundamental domain by a factor of two (e.g. to the region $\text{Re}M \geq 0$). This moduli space with its fixed points and fixed lines is shown in Figure 3. It also shows the generators that act trivially on some of these fixed points and lines. The vertical red lines, for example, correspond to an unbroken \mathcal{CP} transformation. The fixed points of $SL(2, \mathbb{Z})$ are at $M = i$ (stabilized by the generator S) and at $M = \exp(\pi i/3)$ (stabilized by ST). At the fixed points and lines of Figure 3, we obtain enhancements of the flavor group shown in Figure 4. For generic values of the modulus M we have the traditional flavor group $\Delta(54)$. On the lines we have an enhanced group with 108 elements: $[108,17]$. At the points where two lines meet (blue squares) this group is further enhanced to $[216,87]$. The meeting points of three lines (green circles) leads to $[324,39]$, the maximal enhancement in the present example. If one moves away from the fixed points (lines) the symmetries are (partially) spontaneously broken.

7.2. \mathbb{Z}_2 orbifold

Up to now we have concentrated on the two-dimensional \mathbb{Z}_3 orbifold. Let us now add the discussion of the \mathbb{Z}_2 orbifold, as detailed in refs. [35, 36]. It is of interest as it is the simplest example of an orbifold with both unconstrained (complex) moduli of the two-dimensional torus: the Kähler modulus (usually called T , not to be confused with the modular transformation T) and the complex

structure modulus (U). Then, modular transformations form a direct product group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$. This is in contrast to the \mathbb{Z}_3 orbifold, where the complex structure modulus $U = \exp(2\pi i/3)$ needs to be fixed to allow for the \mathbb{Z}_3 twist, breaking $SL(2, \mathbb{Z})_U$ to a remnant R -symmetry. In addition, the traditional flavor symmetry of the \mathbb{Z}_2 orbifold is

$$(D_8 \times D_8)/\mathbb{Z}_2 \cong [32, 49]$$

as a result of the geometry and string selection rules. The two-dimensional \mathbb{Z}_2 orbifold can be understood as the combination of two twisted circles, each contributing D_8 to the traditional flavor group. The eclectic flavor group of the \mathbb{Z}_2 orbifold is completed by the finite modular symmetry which contains the product $\Gamma_2^T \times \Gamma_2^U \cong S_3 \times S_3$ originating from $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$. In addition, the mirror symmetry, which exchanges the two unconstrained moduli T and U , is manifest in the \mathbb{Z}_2 orbifold. It acts as a \mathbb{Z}_2 symmetry on the moduli and as a \mathbb{Z}_4 symmetry on the twisted string modes. This leads to the finite modular group

$$[(S_3^T \times S_3^U) \times \mathbb{Z}_4^{\text{mirror}}] \times \mathbb{Z}_2^{\mathcal{CP}} \cong [288, 880]$$

including the \mathcal{CP} -like symmetry acting as $T \mapsto -\bar{T}$ and $U \mapsto -\bar{U}$ on the moduli. Inclusion of the automorphy factors (with fractional modular weights for matter fields) leads to an additional \mathbb{Z}_4^R R -symmetry. The eclectic flavor group is then the multiplicative closure of

$$(D_8 \times D_8)/\mathbb{Z}_2, \quad (S_3^T \times S_3^U) \times \mathbb{Z}_4^{\text{mirror}}, \quad \mathbb{Z}_4^R \quad \text{and} \quad \mathbb{Z}_2^{\mathcal{CP}},$$

and has a total of 9216 elements.

Again, only part of the eclectic flavor symmetry is linearly realized. At generic values of the moduli we have the traditional flavor symmetry. It is enhanced at various fixed sub-loci of the modular flavor group [288,880] which includes $S_3^T, S_3^U, \mathbb{Z}_4^{\text{mirror}}$ and $\mathbb{Z}_2^{\mathcal{CP}}$. The moduli space has now four real (two complex) dimensions as indicated in Figure 5. There we have defined the curves λ_T and λ_U as the boundaries of the fundamental domains of $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$, respectively. Combined with the fixed domains of mirror symmetry ($T = U$) and the \mathcal{CP} -like symmetry we expect the enhanced unified flavor groups along these lines λ_T and λ_U . The result of the full analysis is shown in Figure 6. The figure illustrates the intriguing interplay of mirror symmetry and finite modular symmetries and we see a variety of local unified flavor groups. The largest group is found at the point $T = U = \exp(i\pi/3)$:

$$([1152, 157463] \times \mathbb{Z}_4^R)/\mathbb{Z}_2.$$

A detailed discussion of Figure 6 can be found in ref. [36].

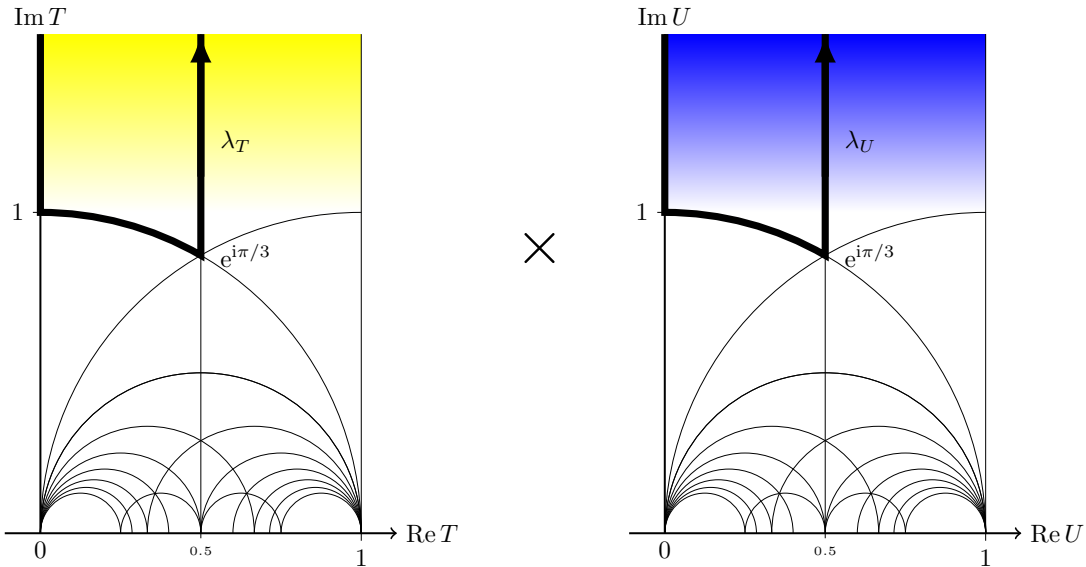


FIGURE 5: The curves λ_T and λ_U are the boundaries of the corresponding fundamental domains. They start at $T = i\infty$, pass by the fixed points and end at $T = 0.5 + i\infty$. It is at these curves where we expect the enhancement of flavor groups.

8. WILSON LINES AND $Sp(4, \mathbb{Z})$

The interplay between mirror symmetry and finite modular symmetries can be made manifest with the consideration of the Siegel modular group $Sp(4, \mathbb{Z})$. From the string theory point of view, it appears as the simplest manifestation of gauge background fields (Wilson lines) on the torus [37]. $Sp(4, \mathbb{Z})$ describes three moduli (T, U and a Wilson line modulus Z) and it contains $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})$ and \mathbb{Z}_2 mirror symmetry as subgroups. For a toroidal string compactification, the generators of $Sp(4, \mathbb{Z})$

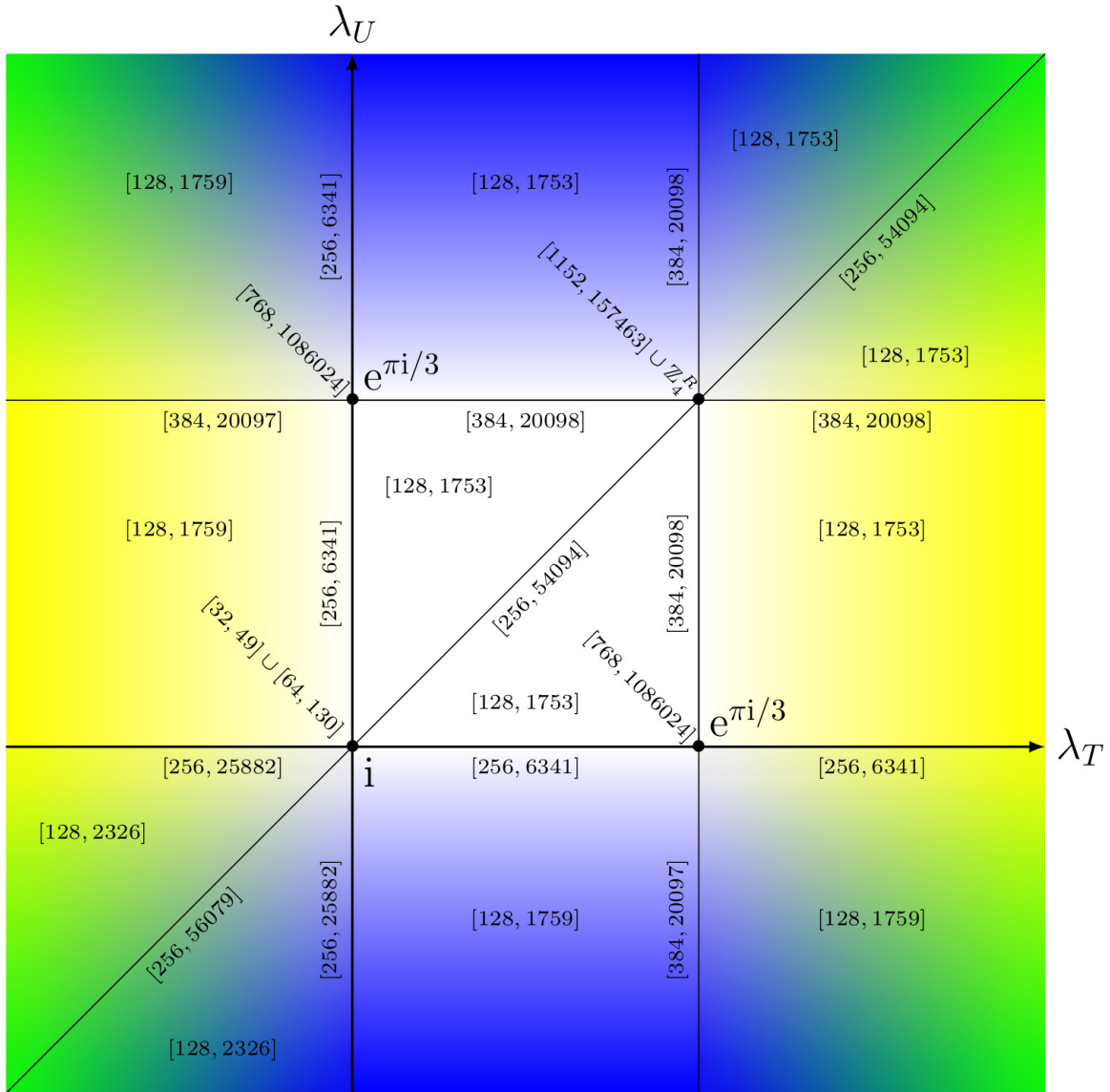


FIGURE 6: Landscape of the unified flavor groups (including \mathcal{CP}) in the λ_T - λ_U plane. We use the abbreviations $[32, 49] \cup [64, 130] \cong [[32, 49] \times [64, 130]]/\mathbb{Z}_2$ and $[1152, 157463] \cup \mathbb{Z}_4^R \cong [[1152, 157463] \times \mathbb{Z}_4^R]/\mathbb{Z}_2$, where the \mathbb{Z}_2 corresponds to the point group selection rule of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, contained in both groups in the product. The axes λ_T and λ_U describe the boundaries of the fundamental domains of T and U (as shown in Figure 5). The diagonal line corresponds to the hypersurface where $U = T$ on the curves λ_T and λ_U . The unified flavor groups above and below the diagonal are related by mirror symmetry \bar{M} .

can be determined via the standard analysis of the outer automorphisms of the Narain space group. An application to the flavor structure of particle physics again requires orbifold twists to obtain chiral fermions. All possible twists are connected to the fixed sub-loci in the fundamental domain (i.e. the Siegel upper half plane) of $\text{Sp}(4, \mathbb{Z})$ [38]. They include the previously considered \mathbb{Z}_K orbifolds ($K = 2, 3, 4, 6$). In addition, a twist by \mathbb{Z}_2 mirror symmetry is included and this twist leads to asymmetric orbifolds, where $\theta_L \neq \theta_R$. From the top-down approach, applications to flavor symmetries have not yet been discussed in detail.

Up to now, there has been one bottom-up attempt of flavor model building with $\text{Sp}(4, \mathbb{Z})$, where the maximal finite modular symmetry was chosen as S_6 [39]. In the case of vanishing third modulus (the Wilson line), S_6 includes and unifies the finite modular groups $(S_3 \times S_3) \rtimes \mathbb{Z}_2$ that appeared earlier in the discussion of the \mathbb{Z}_2 orbifold (where mirror symmetry acts as \mathbb{Z}_2 on the moduli). Hence, also $\text{Sp}(4, \mathbb{Z})$ is expected to include, among others, a generalization of the \mathbb{Z}_2 orbifold. The investigation of the Siegel modular group towards flavor symmetry has just started and we can look forward to many new and exciting aspects of flavor physics. One of them has been found already in the natural appearance of a \mathcal{CP} -like symmetry that extends $\text{Sp}(4, \mathbb{Z})$ to $\text{GSp}(4, \mathbb{Z})$.

9. WHERE ARE WE?

The top-down approach to flavor symmetries predicts the presence of (discrete) traditional and modular flavor symmetries. It unifies these symmetries including \mathcal{CP} within the framework of the eclectic flavor group. While the traditional flavor symmetry is universal in moduli space, there appear non-universal enhancements of flavor symmetries and \mathcal{CP} at specific “points” in moduli space. The spontaneous breakdown of these symmetries can be understood as a motion in moduli space (away from these points or lines of enhancement). If we are close to these “points”, the enhanced symmetries are only slightly broken. The breakdown of the traditional flavor symmetry, however, requires the presence of flavon fields.

This opens up a new arena for flavor model building to be compared to the existing bottom-up constructions. We need more explicit string theory constructions to exhaust the possibilities and to clarify the restrictions. These restrictions do not only concern the possible discrete groups but also the specific representations and modular weights of the matter fields in the low energy sector. At the moment, there is still a huge gap between existing top-down and bottom-up attempts.

As we have seen, string theory includes all the necessary ingredients for a discussion of flavor symmetries. These include:

- **the traditional flavor group,**
- **the finite modular flavor group,**
- **additional R -symmetries (shaping symmetries) from the restrictions of the automorphy factors and the modular weights of matter fields, and**
- **a natural candidate for a \mathcal{CP} -symmetry,**

such that the eclectic flavor group serves as a unified description of the discrete flavor scheme.

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