Constraining gravitational models using cosmological integrability conditions

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Abstract

We present some techniques of constraining gravitational models using the covariant consistency analysis. We will then use the techniques to discuss the integrability conditions of classes of shear-free perfect-fluid cosmological models in both $f(R)$ and scalar-tensor gravitational theories. Among other interesting results, we will show the existence of so-called *anti-Newtonian* universes and universes that rotate and expand simultaneously, both of which are in contrast to the predictions of General Relativity.

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1. INTRODUCTION

$f(R)$ models are a sub-class of 4^{th}-order theories of gravitation, with an action of the form

$$\mathcal{A} = \frac{1}{2}\int d^4x\sqrt{-g}\,[f(R) + 2\mathcal{L}_m] \tag{1}$$

- Simplest generalizations to GR
- An extra degree of freedom
- Cosmological viability
  - observational constraints
  - theoretical constraints: analysis of the integrability conditions on the field equations

The $f(R)$-generalized Einstein field equations can be given by

$$f'G_{ab} = T^m_{ab} + \frac{1}{2}(f - Rf')g_{ab} + \nabla_b\nabla_a f' - g_{ab}\nabla_c\nabla^c f' \tag{2}$$

- Generic viability conditions on f:
  - To ensure gravity remains attractive
  $$f' > 0 \ \forall R$$
  - For stable matter-dominated and high-curvature cosmological regimes (nontachyonic scalaron)
  $$f'' > 0 \ \forall R \gg f''$$
  - GR-like law of gravitation in the early universe (BBN, CMB constraints)
  $$\lim_{R\to\infty}\frac{f(R)}{R} = 1 \Rightarrow f' < 1$$
  - At recent epochs
  $$|f' - 1| \ll 1$$

1.1. *Covariant thermodynamics*

The matter-energy content of the Universe is specified by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_{(a}u_{b)} + \pi_{ab}$$

- Curvature and total fluid thermodynamics

$$\mu_R = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - \Theta f''\dot{R} + f''\tilde{\nabla}^2 R\right]$$
$$p_R = \frac{1}{f'}\left[\frac{1}{2}(f - Rf') + f''\ddot{R} + f'''\dot{R}^2 + \frac{2}{3}\left(\Theta f''\dot{R} - f''\tilde{\nabla}^2 R - f'''\tilde{\nabla}^a R\tilde{\nabla}_a R\right)\right]$$
$$q_a^R = -\frac{1}{f'}\left[f'''\dot{R}\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} - \frac{1}{3}f''\Theta\tilde{\nabla}_a R\right]$$
$$\pi_{ab}^R = \frac{1}{f'}\left[f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R - \sigma_{ab}\dot{R}f''\right]$$

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R\ ,\ \ p \equiv \frac{p_m}{f'} + p_R\ ,\ \ q_a \equiv \frac{q_a^m}{f'} + q_a^R\ ,\ \ \pi_{ab} \equiv \frac{\pi_{ab}^m}{f'} + \pi_{ab}^R$$

The covariant derivative of the timelike vector u^a is decomposed into its irreducible parts as

$$\nabla_a u_b = -A_a u_b + \tfrac{1}{3}h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c$$

$$A_a \equiv \dot{u}_a\ ,\ \ \Theta \equiv \tilde{\nabla}_a u^a\ ,\ \ \sigma_{ab} \equiv \tilde{\nabla}_{\langle a}u_{b\rangle}\ ,\ \ \omega^a \equiv \epsilon^{abc}\tilde{\nabla}_b u_c$$

The trace-free part of the Riemann tensor defines the *Weyl conformal curvature tensor*

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{R}{3}g^{[a}{}_{[c}g^{b]}{}_{d]}$$

- Split into its symmetric, trace-free "electric" and "magnetic" parts, E_{ab} and H_{ab} respectively given by

$$E_{ab} \equiv C_{agbh}u^g u^h, \qquad H_{ab} \equiv \tfrac{1}{2}\eta_{ae}{}^{gh}C_{ghbd}u^e u^d$$

  E_{ab} represents the free gravitational field (tidal forces); H_{ab} is responsible for gravitational waves, no Newtonian analogue

1.2. *Evolution equations*

- $1+3$ covariant splitting of the Bianchi and Ricci identities

$$\nabla_{[a}R_{bc]d}{}^e = 0\ ,\ \ (\nabla_a\nabla_b - \nabla_b\nabla_a)u_c = R_{abc}{}^d u_d$$

  result in propagation and constraint equations

- The evolution equations uniquely determine the covariant variables on some initial hypersurface S_0 at t_0:

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta - \tilde{\nabla}^a q_a^m - 2A_a q_m^a - \sigma_b^a\pi_{a(m)}^b$$
$$\dot{\mu}_R = -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f'^2}\dot{R} - \tilde{\nabla}^a q_a^R - 2A_a q_R^a - \sigma_b^a\pi_{a(R)}^b$$
$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a - A_a A^a - \sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a$$
$$\dot{q}_a^m = -\tfrac{4}{3}\Theta q_a^m - (\mu_m + p_m)A_a - \tilde{\nabla}_a p_m - \tilde{\nabla}^b\pi_{ab}^m - \sigma_a^b q_b^m - A^b\pi_{ab}^m - \epsilon_{abc}\omega^b q_m^c$$

1.3. Evolution equations...

$$
\begin{aligned}
\dot{q}_a^R &= -\tfrac{4}{3}\Theta q_a^R + \frac{\mu_m f''}{f'^2}\tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi_{ab}^R - \sigma_a^b q_b^R \\
&\quad - (\mu_R + p_R)A_a - A^b \pi_{ab}^R - \epsilon_{abc}\omega^b q_R^c \\
\dot{\omega}_a &= -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b A^c + \sigma_a^b \omega_b \qquad (3)\\
\dot{\sigma}_{ab} &= -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a}A_{b\rangle} + A_{\langle a}A_{b\rangle} - \sigma^c_{\langle a}\sigma_{b\rangle c} \\
&\quad - \omega_{\langle a}\omega_{b\rangle} \qquad (4)\\
\dot{E}_{ab} + \tfrac{1}{2}\dot{\pi}_{ab} &= \epsilon_{cd\langle a}\tilde{\nabla}^c H^d_{b\rangle} - \Theta\left(E_{ab} + \tfrac{1}{6}\pi_{ab}\right) - \tfrac{1}{2}(\mu + p)\sigma_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q_{b\rangle} \\
&\quad + 3\sigma_a^{\langle c}\left(E_{b\rangle c} - \tfrac{1}{6}\pi_{b\rangle c}\right) - A_{\langle a}q_{b\rangle} + \epsilon_{cd\langle a}\left[2A^c H^d_{b\rangle} + \omega^c(E^d_{b\rangle} + \tfrac{1}{2}\pi^d_{b\rangle})\right] \\
&\qquad (5)\\
\dot{H}_{ab} &= -\Theta H_{ab} - \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} + \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi^d_{b\rangle} \\
&\quad + 3\sigma_a^{\langle c}H_{b\rangle c} + \tfrac{3}{2}\omega_{\langle a}q_{b\rangle} - \epsilon_{cd\langle a}\left[2A^c E^d_{b\rangle} - \tfrac{1}{2}\sigma^c_{b\rangle}q^d - \omega^c H^d_{b\rangle}\right] \qquad (6)
\end{aligned}
$$

1.4. Constraints

- Restrict the initial data to be specified; must remain satisfied on any hypersurface S_t for all t

$$
\begin{aligned}
(C^1)_a &:= \tilde{\nabla}^b\sigma_{ab} - \tfrac{2}{3}\tilde{\nabla}_a\Theta + \epsilon_{abc}\left(\tilde{\nabla}^b\omega^c + 2A^b\omega^c\right) + q_a = 0 \\
(C^2)_{ab} &:= \epsilon_{cd(a}\tilde{\nabla}^c\sigma_{b)}{}^d + \tilde{\nabla}_{\langle a}\omega_{b\rangle} - H_{ab} - 2A_{\langle a}\omega_{b\rangle} = 0 \qquad (7)\\
(C^3)_a &:= \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + \epsilon_{abc}\left[\tfrac{1}{2}\tilde{\nabla}^b q^c + \sigma_{bd}\left(E^d{}_c + \tfrac{1}{2}\pi^d{}_c\right)\right] \\
&\quad + 3\omega_b\left(E^{ab} - \tfrac{1}{6}\pi^{ab}\right) = 0 \\
(C^4)_a &:= \tilde{\nabla}^b E_{ab} + \tfrac{1}{2}\tilde{\nabla}^b\pi_{ab} - \tfrac{1}{3}\tilde{\nabla}_a\mu + \tfrac{1}{3}\Theta q_a \\
&\quad - \tfrac{1}{2}\sigma_a^b q_b - 3\omega^b H_{ab} - \epsilon_{abc}[\sigma^{bd}H^c_d - \tfrac{3}{2}\omega^b q^c] = 0 \\
(C^5) &:= \tilde{\nabla}^a\omega_a - A_a\omega^a = 0 \qquad (8)
\end{aligned}
$$

- The Gauß-Codazzi equations are given by

$$\tilde{R}_{ab} + \dot{\sigma}_{\langle ab\rangle} + \Theta\sigma_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} - A_{\langle a}A_{b\rangle} - \pi_{ab} - \frac{1}{3}\left(2\mu - \frac{2}{3}\Theta^2\right)h_{ab} = 0 \qquad (9)$$

2. SIMULTANEOUSLY ROTATING AND EXPANDING MODELS

Classic GR result (Gödel, Ellis): shear-free perfect-fluid cosmological models (homogeneous, inhomogeneous) cannot rotate and expand simultaneously, *i.e.,*

$$\Theta\omega^a = 0$$

- Turning off the shear from the propagation equations results in a new constraint equation

$$(C^6)_{ab} := E_{ab} - \tfrac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} = 0$$

- Demanding consistent spatial (curl) and temporal (time derivative) propagations results in [1]

$$
\begin{aligned}
&\Theta\omega^a\Bigg\{\left[\frac{(1-w)P}{3}\tilde{R} + \frac{(1+w)}{f'}\frac{(3w+5)f' + 4f''Q}{6f'}\mu_m\right] \\
&+\frac{Z}{P}\left[(\frac{1+w}{f'})\mu_m\right]\Bigg\} = 0 \qquad (10)
\end{aligned}
$$

2.1. Flat, vacuum solutions

In the above result, we have defined

$$
\begin{aligned}
&\Theta \equiv 3\frac{\dot{a}}{a}, \quad q \equiv -\frac{\ddot{a}a}{\dot{a}^2}, \quad j \equiv \frac{\dddot{a}\,a^2}{\dot{a}^3}, \quad s \equiv \frac{a^3}{\dot{a}^4}\frac{d^4a}{dt^4} \\
&Q \equiv \frac{1}{3}\Theta^2(j-q-2)+\tilde{R} \\
&P \equiv \frac{f''}{f'}Q+\frac{3w}{2} \\
&Z \equiv \frac{2}{3}\left(\frac{f'''}{f'}-(\frac{f''}{f'})^2\right)Q^2+\frac{f''}{9f'}\left((4+5q+j+jq+s)\Theta^2+6\tilde{R}\right)
\end{aligned}
$$

- It follows that we must have either $\omega^a\Theta = 0$ or the expression in the curly brackets of Eq. (10) must vanish
- Notice that if the 3-curvature vanishes $\tilde{R}$, then the GR result can always be avoided for vacuum universes ($\mu_m = 0$), *i.e.*, a shear-free, spatially flat vacuum universe in any $f(R)$ theory can rotate and expand simultaneously in the linearized regime

2.2. Non-vacuum, Milne solutions

- For the non-vacuum case, it can be shown that using flat Milne universe solutions

$$\mu_m = \frac{\mu_0}{a^{3(1+w)}}\ , \quad \dot{\Theta} = -\tfrac{1}{3}\Theta^2, \quad R = \tfrac{2}{3}\Theta^2\ , \quad a(R) = \frac{1}{\sqrt{R}}$$

into the Friedmann equation

$$\frac{1}{3}\Theta^2 = \frac{1}{f'}\left[\mu_m + \frac{Rf'-f}{2} - \Theta\dot{R}f''\right]\ ,$$

one gets

$$-R^2\frac{d^2f(R)}{dR^2}+\frac{f(R)}{2}-\frac{\mu_0}{a(R)^{3(1+w)}} = 0\ ,$$

which has the following general solution:

$$f(R) = C_1R^{\frac{1+\sqrt{3}}{2}} + C_2R^{\frac{1-\sqrt{3}}{2}} - \frac{4\mu_0}{1+12w+9w^2}R^{\frac{3(1+w)}{2}} \tag{11}$$

If we consider the R^n toy model, the term in the curly brackets of Eq. (10) reduces to

$$\frac{(1+w)\mu_m}{6f'}\left[3w+9-4n\right] = 0 \tag{12}$$

- Comparing solutions (12) and the particular solution of Eq. (11), we get $w = 1$ if $\mu_m \neq 0$, *i.e.*, for a stiff fluid in R^3 gravity, there exists a flat Milne-universe solution which can rotate and expand simultaneously at the level of linearised perturbation theory
- This suggests that there are situations where linearized fourth-order gravity shares properties with Newtonian theory not valid in GR

3. IRROTATIONAL MODELS

For classes of non-rotating fluid models, the vorticity vanishes: $\omega_a = 0$ will have the evolution equation (3) turned into a new constraint

$$(C^{6*})_a := \epsilon_{abc}\tilde{\nabla}^bA^c = 0 \implies A_a = \tilde{\nabla}_a\psi$$

for some scalar ψ. Taking the curl and temporal derivative of this constraint results in the mathematical identities

$$
\begin{aligned}
&\left(\epsilon_{abc}\tilde{\nabla}^bA^c\right)^{\cdot} = 0 \\
&curl(curl(A_a)) = \tilde{\nabla}_a\left(\tilde{\nabla}^2\psi\right) - \tilde{\nabla}^2\left(\tilde{\nabla}_a\psi\right) + \tfrac{2}{3}\left(\mu - \tfrac{1}{3}\Theta^2\right)\tilde{\nabla}_a\psi = 0
\end{aligned}
$$

- Generic irrotational fluid models in $f(R)$ gravity are self-consistent [2]!

3.1. Dust models

On the other hand, if we specialize to dust models

$$w = 0 = p_m\ , \quad q^m_a = 0 = A_a\ , \quad \pi^m_{ab} = 0\ ,$$

then some interesting integrability conditions arise. For example, in shear-free dust models, *i.e.*, $\sigma_{ab} = 0$, Eq. (4) turns into a new constraint

$$(C^{6d})_{ab} := E_{ab} - \tfrac{1}{2}\pi^R_{ab} = 0 \tag{13}$$

- Unlike in GR, E_{ab} does not vanish because π^R_{ab} is nonzero, but H_{ab} does vanish, leading to a modified constraint from Eq. (6), which obviously is an identity:

$$(C^{7d})_{ab} := \epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} - \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi^{d\ d}_{b\rangle} = 0$$

- Since q^R_a becomes irrotational, it can be shown that for some scalar field ϕ and some spatially constant scalar C:

$$q^R_a = \tilde{\nabla}_a\phi\ , \quad \phi = \tfrac{2}{3}\Theta + C \tag{14}$$

An interesting consequence of the above result is the integrability condition

$$\tfrac{2}{3}f'\tilde{\nabla}_a\Theta + \left(f'''\dot{R} - \tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_a R + f''\tilde{\nabla}_a\dot{R} = 0$$

- In the GR limit, we get a spatially homogeneous expansion

$$\tilde{\nabla}_a\Theta = 0$$

- Propagating the new constraint above results in the new equation

$$\dot{\pi}^R_{ab} + \tfrac{2}{3}\Theta\pi^R_{ab} - \tfrac{1}{2}\tilde{\nabla}_{\langle a}q^R_{b\rangle} = 0\ ,$$

implying that irrotational shear-free dust spacetimes governed by $f(R)$ gravitational physics evolve consistently if

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + \tfrac{3f''}{2f'}\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R} = 0 \tag{15}$$

- The GR limit of the above equation is an identity since the left-hand side vanishes identically

Now since for any scalar field ψ,

$$\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\psi = \epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{(b}\tilde{\nabla}^{d)}\psi = \epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_b\tilde{\nabla}^d\psi = 0$$

taking the curl of Eq. (15) results in another identity:

$$\left[\frac{3}{2}\left(\frac{f'''}{f'} - \frac{f''^2}{f'^2}\right)\dot{R} - \frac{\Theta f''}{6f'}\right]\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}R + \tfrac{3f''}{2f'}\epsilon_{cda}\tilde{\nabla}^c\tilde{\nabla}_{\langle b}\tilde{\nabla}^{d\rangle}\dot{R} = 0 \tag{16}$$

- This suggests that all irrotational shear-free dust spacetimes in $f(R)$-gravity are self-consistent

- For the conformally flat metric, *i.e.*, if $E_{ab} = 0$ as well, the following new linearized constraints emerge:

$$\begin{aligned}\tilde{\nabla}_{\langle a}q^R_{b\rangle} &= 0 = \left(\dot{R}f''' - \tfrac{1}{3}\Theta f''\right)\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\dot{R}\\ \pi^R_{ab} &= 0 = f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R\end{aligned}$$

3.2. Dust spacetimes with div $H_{ab} = 0$

3.3. Irrotational dust spacetimes with div $H_{ab} = 0$

A necessary condition for the propagation of gravitational waves is the vanishing of the divergence of a non-zero H_{ab}.

- Prescribing this condition on the field equations results in a generalized constraint of the irrotational q^R_a term we saw a few slides back, Eq. (14):

$$\tilde{\nabla}_a\phi = \tfrac{2}{3}\tilde{\nabla}_a\Theta - \tilde{\nabla}^b\sigma_{ab} \tag{17}$$

- A subclass of such models, called "purely radiative" dust spacetimes, is a divergence-free E_{ab}. Such models in $f(R)$ gravity are constrained further as

$$\tilde{\nabla}_a\mu_m + f'\tilde{\nabla}_a\mu_R + f'\Theta q^R_a - \tfrac{3f'}{2}\tilde{\nabla}^b\pi^R_{ab} = 0 \tag{18}$$

  - In GR purely radiative irrotational dust spacetimes are spatially homogeneous:

$$\tilde{\nabla}_a\mu_m = 0 \tag{19}$$

3.4. Non-expanding spacetimes

Here we want to explore the (in)consistencies that emerge assuming theoretical cases of a non-expanding spacetime, i.e., $\Theta = 0$.

- One can immediately conclude, for example, that a new constraint arises from the Raychaudhuri equation:

$$(C^{6s}) := \tilde{\nabla}_a A^a - \tfrac{1}{2f'}(1+3w)\mu_m - \tfrac{1}{2}\left(\mu_R + 3p_R\right) = 0 \tag{20}$$

For dust models ($A_a = 0 = q^m_a$), this would mean a vanishing active gravitational mass: $\mu + 3p = 0$. Furthermore, the conservation equation would guarantee that $\mu_d(t) = \text{const}$, and hence that $\mu_R + 3p_R = \text{const}$, as well. Combining this with the *trace equation*

$$3f''\ddot{R} + 3\dot{R}^2 f''' + 3\Theta\dot{R}f'' - 3f''\tilde{\nabla}^2 R - Rf' + 2f - \mu_m + 3p_m = 0$$

we conclude that

$$f - 2f''\tilde{\nabla}^2 R = \text{const} \tag{21}$$

- Any non-rotating, non-expanding dust spacetime in $f(R)$ cosmology should have a gravitational Lagrangian that satisfies Eq. (21)

4. QUASI-NEWTONIAN MODELS

- Irrotational dust universes with purely gravito-magnetic Weyl tensor $\longrightarrow$ quasi-Newtonian universes, characterized by

$$p_m = 0\,, \quad A_a = 0\,, \quad q^m_a = \mu_m v_a\,, \quad \pi^m_{ab} = 0\,, \quad \omega_a = 0\,, \quad H_{ab} = 0$$

  - potential models for the description of gravitational collapse and late-time cosmic structure

- Choose a comoving 4-velocity $\tilde{u}^a$ such that

$$\tilde{u}^a = u^a + v^a\,, \quad v_a u^a = 0\,, \quad v_a v^a << 1\,,$$

where v^a is the non-relativistic ("peculiar") velocity and vanishes in the background

For this class of models, it can be shown that

$$\frac{1}{2}\epsilon^{abc}\tilde{\nabla}_b A_c = 0 \implies A_a \equiv \tilde{\nabla}_a \Phi$$
$$E_{ab} - \frac{1}{2}\pi_{ab} - \tilde{\nabla}_{\langle a}A_{b\rangle} = 0$$

For any fourth-order gravity model in which the anisotropic pressure π_{ab} can be given in terms of a scalar potential Ψ as [3]

$$\pi_{ab} = \tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\Psi$$

- Two generally independent integrability conditions for generic fluid models exist:

$$\tilde{\nabla}_{<a}\tilde{\nabla}_{b>}\left(\dot{\Phi} + \frac{1}{3}\Theta + \dot{\Psi}\right) + \left(\dot{\Phi} + \frac{1}{3}\Theta + \dot{\Psi}\right)\tilde{\nabla}_{<a}\tilde{\nabla}_{b>}\Phi = 0 \tag{22}$$

$$\begin{aligned} &6\tilde{\nabla}_a\ddot{\Phi} + 6\Theta\tilde{\nabla}_a\dot{\Phi} - \left(2\mu - \frac{2}{3}\Theta^2\right)\tilde{\nabla}_a\Phi + 6\tilde{\nabla}_a\ddot{\Psi} + 6\Theta\tilde{\nabla}_a\dot{\Psi} \\ &- \left(2\mu - \frac{2}{3}\Theta^2\right)\tilde{\nabla}_a\Psi - 2\tilde{\nabla}_a(\tilde{\nabla}^2\Psi) - 3\tilde{\nabla}_a p = 0 \end{aligned} \tag{23}$$

- Identically the same in $f(R)$ models, due to the linearized form of π^R_{ab} in Eq. (3)

- Modified Poisson equation

$$\tilde{\nabla}^2\Phi = \frac{1}{2}\left(\mu + 3p\right) - \left[3\left(\ddot{\Phi} + \ddot{\Psi}\right) + \left(\dot{\Phi} + \dot{\Psi}\right)\Theta\right]$$

- Velocity perturbations are scale-independent, as in GR, but matter density fluctuations are scale-dependent

- Over regions of space-time where the Ricci curvature scalar is a slowly varying function of space and time
  - $f(R)$ (and its derivatives) are associated Laguerre polynomials
  - The peculiar velocity, 4-acceleration, total cosmic heat flux and anisotropic stress can be analytically calculated explicitly

5. ANTI-NEWTONIAN MODELS

- Irrotational dust universes with purely gravito-magnetic Weyl tensor $\longrightarrow$ anti-Newtonian universes, characterized by

$$p_m = 0\,, \quad A_a = 0\,, \quad q_a^m = 0\,, \quad \pi_{ab}^m = 0\,, \quad \omega_a = 0\,, \quad E_{ab} = 0$$

  - Farthest possible models from Newtonian universes

- In GR, anti-Newtonian universes suffer from severe integrability conditions, no known anti-Newtonian spacetimes that are linearized perturbations of Friedman-Lemaître-Robertson-Walker (FLRW) universes

- In fourth-order gravitational theories, anti-Newtonian models exist, subject to the integrability condition [4]

$$\tilde{\nabla}^2 q_a^R - \tilde{\nabla}_a(\tilde{\nabla}^b q_b^R) + \tilde{R} q_a^R + \frac{4f''}{f'^2}\mu_m \Theta \tilde{\nabla}_a R = 0 \tag{24}$$

- For flat universes ($K = 0 = \tilde{R}$) this holds only if

$$f''\mu_m \Theta \tilde{\nabla}_a R = 0 \tag{25}$$

  - Impose $\mu_m \neq 0$ and $f'' \neq 0$. For a consistently evolving set of constraints in the flat, anti-Newtonian spacetimes, either one of the following conditions must hold:

$$\Theta = 0 \quad \longrightarrow \text{static}$$
$$\tilde{\nabla}_a R = 0 \quad \longrightarrow \text{homogeneous}$$

- Closed & open universes ($K = \pm 1$): any dust solution of

$$\left[\frac{f''\mu_m\Theta}{f'} \mp \frac{2}{a^2}\left(\dot{R}f''' - \tfrac{1}{3}\Theta f''\right)\right]\tilde{\nabla}_a R \mp \frac{2f''}{a^2}\tilde{\nabla}_a \dot{R} = 0 \tag{26}$$

  with $f'' \neq 0$ is an anti-Newtonian solution

6. SHEAR-FREE ANISOTROPIC MODELS

- In orthogonal models with irrotational and non-accelerated fluid flows without heat fluxes:

$$T_{ab}^m = \mu_m u_a u_b + p_m h_{ab} + \pi_{ab}^m\,, \qquad \omega_a = 0 = A_a$$

- From causal relativistic thermodynamical relationships for imperfect fluids, the anisotropic pressure is known to evolve according to

$$\tau \dot{\pi}_{ab} + \pi_{ab} = -\lambda \sigma_{ab} \tag{27}$$

  - τ and λ are relaxation and viscosity parameters

- For negligible τ and positive constant λ; Ansatz for the equation of state:

$$\pi_{ab} = -\lambda \sigma_{ab} \tag{28}$$

  - Valid near thermal equilibrium, such as in the very early stages of the Universe

- Eqs. (3) imply that we can rewrite (28) as [5]

$$\pi_{ab}^m + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle} R + f'''\tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b\rangle} R = \sigma_{ab}\left(\dot{R}f'' - \lambda f'\right) \tag{29}$$

- For a general case of vanishing shear tensor during the entire cosmic evolution, one can see from Eq. (29) that

$$\pi_{ab}^m = -f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle} R - f'''\tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b\rangle} R$$

- The Gauß-Codazzi equations (9) reduce to

$$\tilde{R}_{ab} - \tfrac{1}{3}\tilde{R} h_{ab} = \pi_{ab} = \frac{1}{f'}\left(\pi_{ab}^m + f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle} R + f'''\tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b\rangle} R\right)$$

  Even if the matter anisotropic stress vanishes, no constant-curvature geometrie are guaranteed and hence no necessarily FLRW universes

- Unlike in GR, if we allow the matter anisotropic pressure to be nonzero despite a vanishing shear, constant-curvature models are allowed provided
$$f''\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}R + f'''\tilde{\nabla}_{\langle a}R\tilde{\nabla}_{b\rangle}R = 0$$
- One can see the tidal effect on the anisotropic stresses by dropping the shear terms of Eq. (4), obtaining the equation
$$\pi_{ab} = 2E_{ab} \tag{30}$$
The anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other, a necessary and sufficient condition for the shear to remain zero if initially vanishing
- For nonzero, but very small (second-order) shear, one can show that Eq. (4) can be approximated by
$$\dot{\sigma}_{ab} \approx -\tfrac{2}{3}\Theta\sigma_{ab} \implies \left(\sigma^2\right)^{\cdot} \approx -\tfrac{4}{3}\Theta\sigma^2 \ ,$$
showing that the shear decays with expansion. Within the class of orthogonal $f(R)$ models, small perturbations of shear are damped, *i.e.*, that these models are stable if expanding
- For shear-free orthogonal models satisfying Eq. (30), Eq. (6) reduces to an identity:
$$\epsilon_{cd\langle a}\tilde{\nabla}^c E^d_{b\rangle} = \tfrac{1}{2}\epsilon_{cd\langle a}\tilde{\nabla}^c \pi^d_{b\rangle}$$
- It is straightforward to show using Eqs.(5) and (8) that
$$\begin{aligned}\dot{E}_{ab} &= -\tfrac{2}{3}\Theta E_{ab} - \tfrac{1}{4}\tilde{\nabla}_{\langle a}q^R_{b\rangle}\\ \tilde{\nabla}^b E_{ab} &= \tfrac{1}{6}\left(\tilde{\nabla}_a\mu - \tfrac{1}{3}\Theta q^R_a\right)\end{aligned} \tag{31}$$
- Defining $E^2 \equiv E_{ab}E^{ab}$, and rewriting Eq. (31) as
$$\left(E^2\right)^{\cdot} = -\tfrac{4}{3}\Theta E^2 - \frac{1}{8}\left(\tilde{\nabla}_{\langle a}q^R_{b\rangle}E^{ab} + \tilde{\nabla}^{\langle a}q^{b\rangle}_R E_{ab}\right) \tag{32}$$
shows the decay of the electric part of the Weyl tensor and the anisotropic stress tensor with expansion
- Since the generalized Friedman equation does not guarantee a positive total energy density,
$$\Theta^2 = 3\left(\mu - \tfrac{1}{2}\tilde{R}\right) \ ,$$
it is not straightforward to comment on the asymptotic isotropization of expanding shear-free anisotropic models for the different values of the spatial curvature
- This is in contrast to the GR result where, for example, expanding shear-free models which exhibit negative spatial curvature asymptotically approach isotropy

7. CONCLUSIONS

- In summary, we have
  - looked at the consistency relations of linearized perturbations of FLRW universes arising as a result of imposing special restrictions to the field equations in $f(R)$ gravity
  - shown that, contrary to the results of GR, simultaneously rotating and expanding spacetimes exist in modified gravity
  - explored different classes of non-rotating fluid models in $f(R)$ gravity, and their corresponding GR implications
  - briefly discussed the existence of integrability conditions for Newtonian-like and anti-Newtonian cosmological models
  - studied the $f(R)$-gravity dynamics of shear-free anisotropic cosmologies vis-à-vis general relativistic physics
- The important point here is that, even at the theoretical level, if the exact dynamical evolution of the Universe is known, one can, *in principle*, constrain the gravitational action for the underlying physics

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